Higher Order Domination of Graphs

Stephen Benecke

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Supervisor: Prof JH van Vuuren
Co–supervisor: Dr PJP Grobler

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signature: ____________________ Date: ____________________

Pectora roborant cultus recti

Abstract

Motivation for the study of protection strategies for graphs is rooted in antiquity and has evolved as a subdiscipline of graph theory since the early 1990s. Using, as a point of departure, the notions of *weak Roman domination* and *secure domination* (where protection of a graph is required against a single attack) an initial framework for higher order domination was introduced in 2002 (allowing for the protection of a graph against an arbitrary finite, or even infinite, number of attacks). In this thesis, the theory of higher order domination in graphs is broadened yet further to include the possibility of an arbitrary number of guards being stationed at a vertex.

The thesis firstly provides a comprehensive survey of the combinatorial literature on *Roman domination*, *weak Roman domination*, *secure domination* and other higher order domination strategies, with a view to summarise the state of the art in the theory of higher order graph domination as at the start of 2004.

Secondly, a generalised framework for higher order domination is introduced in two parts: the first catering for the protection of a graph against a *finite* number of consecutive attacks, and the second concerning the perpetual security of a graph (protection of the graph against an *infinite* number of consecutive attacks). Two types of higher order domination are distinguished: *smart* domination (requiring the *existence* of a protection strategy for any sequence of consecutive attacks of a pre–specified length, but leaving it up to a strategist to uncover such a guard movement strategy for a particular instance of the attack sequence), and *foolproof* domination (requiring that *any* possible guard movement strategy be a successful protection strategy for the graph in question). Properties of these higher order domination parameters are examined — first by investigating the application of known higher order domination results from the literature, and secondly by obtaining new results, thereby hopefully improving current understanding of these domination parameters.

Thirdly, the thesis contributes by (i) establishing higher order domination parameter values for some special graph classes not previously considered (such as complete multipartite graphs, wheels, caterpillars and spiders), by (ii) summarising parameter values for special graph classes previously established (such as those for paths, cycles and selected cartesian products), and by (iii) improving higher order domination parameter bounds previously obtained (in the case of the cartesian product of two cycles).

Finally, a clear indication of unresolved problems in higher order graph domination is provided in the conclusion to this thesis, together with some suggestions as to possibly desirable future generalisations of the theory.

Opsomming

Die motivering vir die studie van verdedigingstrategieë vir grafieke het sy ontstaan in die antieke wêreld en het sedert die vroeë 1990s as 'n subdissipline in grafiekteorie begin ontwikkel. Deur gebruik te maak van die idee van *swak Romynse dominasie* en *versterkte dominasie* (waar verdediging van 'n grafiek teen 'n enkele aanval vereis word) het 'n aanvangsraamwerk vir hoër–orde dominasie (wat 'n grafiek teen 'n veelvuldige, of selfs oneindige aantal, aanvalle verdedig) in 2002 die lig gesien. Die teorie van hoër–orde dominasie in grafieke word in hierdie tesis verbreed, deur toe te laat dat 'n arbitrêre aantal wagte by elke punt van die grafiek gestasioneer mag word.

Eerstens voorsien die tesis 'n omvangryke oorsig van die kombinatoriese literatuur oor *Romynse dominasie*, *swak Romynse dominasie*, *versterkte dominasie* en ander hoër–orde dominasie strategieë, met die doel om die kundigheid betreffende die teorie van hoër–orde dominasie, soos aan die begin van 2004, op te som.

Tweedens word 'n veralgemeende raamwerk vir hoër–orde dominasie bekendgestel, en wel in twee dele. Die eerste deel maak voorsiening vir die verdediging van 'n grafiek teen 'n *eindige* aantal opeenvolgende aanvalle, terwyl die tweede deel betrekking het op die oneindige sekuriteit van 'n grafiek (verdediging teen 'n *oneindige* aantal opeenvolgende aanvalle). Daar word tussen twee tipes höer–orde dominasie onderskei: *intelligente* dominasie (wat slegs die *bestaan* van 'n verdedigingstrategie vir enige reeks opeenvolgende aanvalle vereis, maar dit aan 'n strateeg oorlaat om 'n suksesvolle bewegingstrategie vir die verdediging teen 'n spesifieke reeks aanvalle te vind), en *onfeilbare* dominasie (wat vereis dat *enige* moontlike bewegingstrategie resulteer in 'n suksesvolle verdedigingstrategie vir die betrokke grafiek). Eienskappe van hierdie hoër–orde dominasie parameters word ondersoek, deur eerstens die toepasbaarheid van bekende hoër–orde dominasie resultate vanuit die literatuur te assimileer, en tweedens nuwe resultate te bekom, in die hoop om die huidige kundigheid met betrekking tot hierdie dominasie parameters te verbreed.

Derdens word 'n bydrae gelewer deur (i) hoër–orde dominasie parameterwaardes vas te stel vir sommige spesiale klasse grafieke wat nie voorheen ondersoek is nie (soos volledig veelledige grafieke, wiele, ruspers en spinnekoppe), deur (ii) parameterwaardes wat reeds bepaal is (soos byvoorbeeld dié vir paaie, siklusse en sommige kartesiese produkte) op te som, en deur (iii) bekende hoër–orde dominasie parametergrense te verbeter (in die geval van die kartesiese produk van twee siklusse).

Laastens word 'n aanduiding van oop probleme in die teorie van hoër–orde dominasie in die slothoofstuk van die tesis voorsien, tesame met voorstelle ten opsigte van moontlik sinvolle veralgemenings van die teorie.

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Terms of Reference

After reading the paper "Defend the Roman Empire!" by Ian Stewart [27], Ernie Cockayne (School of Mathematics and Statistics, University of Victoria) and Stephen & Sandee Hedetniemi (Department of Computer Science, Clemson University) decided to study the topic of Roman domination, using a graph theoretic approach. Paul Dreyer (RAND Corporation, Santa Monica, then at Rutgers University) was also involved in this research through his Ph.D. dissertation on the topic. The paper "Roman domination in graphs" [6] was a product of this collaboration.

Stephen Hedetniemi presented a principal lecture on Roman domination at the *Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms, and Applications* held at Western Michigan University in June 2000. In his talk, he posed the problem of characterising Roman trees. Michael Henning (School of Mathematics, Statistics and Information Technology, University of KwaZulu–Natal, Pietermaritzburg), who was present at this lecture, started working on this problem, which resulted in the paper "A Characterization of Roman Trees" [16]. Stephen Hedetniemi also discussed the possibility of a more efficient graph protection model with Michael Henning. Research on this notion resulted in their joint paper "Defending the Roman Empire – A new strategy" [17].

The next generalisation in this area of domination theory was due to Ernie Cockayne. In January 2002, Kieka Mynhardt (School of Mathematics and Statistics, University of Victoria, then at Department of Mathematics, Applied Mathematics and Astronomy, UNISA) organised a workshop on selected graph theoretic topics, called the *Graph Theory Concentration Camp* at the Sunnyside Campus of UNISA. Ernie Cockayne presented a talk at this workshop on Roman domination, weak Roman domination, and generalisations of these notions. Participating in the group session on this topic was Jan van Vuuren (Department of Applied Mathematics, University of Stellenbosch), Paul Grobler (Department of Applied Mathematics, University of Stellenbosch, then at School of Mathematical Sciences, University of Natal, Durban), Justin Munganga (Department of Mathematics, Applied Mathematics and Astronomy, UNISA) and Ken Halland (Department of Computer Science, UNISA), among others. The work done during the session gave rise to the paper "Protection of a Graph" [8]. A direct consequence of this workshop was a visit by Alewyn Burger (School of Mathematics and Statistics, University of Victoria, then at Department of Mathematics, Applied Mathematics and Astronomy, UNISA), Ernie Cockayne, Odile Favaron (Laboratoire de Recherche en Informatique, Université Paris Sud) and Kieka Mynhardt to the Department of Applied Mathematics at the University of Stellenbosch in April 2002. As a result of a suggestion by Ernie Cockayne during this visit, Jan van Vuuren and Alewyn Burger, as well as two graduate students at the Department of Applied Mathematics at the University of Stellenbosch, Werner Gründlingh and Wynand Winterbach, initiated a research project on the problem now known as *higher order domination in graphs*. This research project culminated in the two papers "Finite Order Domination in Graphs" [2] and "Infinite Order Domination in Graphs" [3].

Meanwhile, independent from these events, Michael Henning presented a talk on weak Roman domination at Rand Afrikaans University early in 2002. After his talk, Elna Ungerer (Department of Mathematics, Rand Afrikaans University) suggested the problem of defending the Roman Empire from multiple attacks. This lead to his paper "Defending the Roman Empire from multiple attacks" [18]. Around the same time, Stephen & Sandee Hedetniemi and Wayne Goddard (Department of Computer Science, Clemson University) considered it a natural generalisation to consider the ideas of eternal and mobile security. Research on this generalisation resulted in the paper "Eternal Security in Graphs" [12].

This thesis serves to provide a comprehensive survey of the known results on, and the general state of the art in the topic of higher order domination in 2004. The topic of the thesis was suggested to the author by Jan van Vuuren, who was also the supervisor for this study, while Paul Grobler acted as co–supervisor. It is acknowledged that Alewyn Burger contributed valuable proof suggestions at certain stages during the course of the study. The thesis was commenced in February 2003 and completed in September 2004. Work emanating from the study was presented as papers at both the 2003 and 2004 annual conferences of the South African Mathematical Society in Johannesburg and Potchefstroom, as well as the 2004 annual conference of the Operations Research Society of South Africa in Bellville, and also resulted in the paper "Protection of Complete Multipartite Graphs" [1], submitted in May 2004 for possible publication in Utilitas Mathematica.

Pectora roborant cultus recti

Reserved Symbols

$\beta(G)$	The independence number of a graph G.
C_n	A cycle of order n.
$C(p_1, p_2, \ldots, p_n)$	A caterpillar with p_i leaves joined to the ith vertex of the path P_n, $i = 1, 2, \ldots, n$.
$\mathfrak{c}(G)$	The clique partition number of a graph G.
$\chi(G)$	The (vertex) chromatic number of a graph G.
$\deg_G v$	The degree of a vertex v in a graph G.
$\Delta(G)$	The maximum vertex degree of a graph G.
$\delta(G)$	The minimum vertex degree of a graph G.
$E(G)$	The edge set of a graph G.
$\mathrm{epn}(v, S)$	The set of all S–external private neighbours of a vertex v.
f	A guard function of a graph.
G	A graph $G = (V, E)$, with vertex set V and edge set E.
$\overline{G}$	The complement of the graph G.
$\Gamma(G)$	The upper domination number of a graph G.
$\gamma(G)$	The (lower) domination number of a graph G.
$\gamma_{\ell,k}(G)$	The smart k^{th}–order ℓ–domination number of a graph G.
$\gamma_{\ell,\infty}(G)$	The smart ∞–order ℓ–domination number of a graph G.
$\gamma^*_{\ell,k}(G)$	The foolproof k^{th}–order ℓ–domination number of a graph G.
$\gamma^*_{\ell,\infty}(G)$	The foolproof ∞–order ℓ–domination number of a graph G.
$\gamma_R(G)$	The Roman domination number of a graph G.
$\gamma^k_R(G)$	The k–Roman domination number of a graph G.
$\gamma_r(G)$	The smart weak Roman domination number of a graph G.
$\gamma^*_r(G)$	The foolproof weak Roman domination number of a graph G.
$\gamma_{r,k}(G)$	The smart k–weak Roman domination number of a graph G.
$\gamma_{r,\infty}(G)$	The smart ∞–weak Roman domination number of a graph G.
$\gamma^*_{r,k}(G)$	The foolproof k–weak Roman domination number of a graph G.
$\gamma^*_{r,\infty}(G)$	The foolproof ∞–weak Roman domination number of a graph G.
$\gamma_s(G)$	The smart secure domination number of a graph G.
$\gamma^*_s(G)$	The foolproof secure domination number of a graph G.
$\gamma_{s,k}(G)$	The smart k–secure domination number of a graph G.
$\gamma_{s,\infty}(G)$	The smart ∞–secure domination number of a graph G.
$\gamma^*_{s,k}(G)$	The foolproof k–secure domination number of a graph G.
$\gamma^*_{s,\infty}(G)$	The foolproof ∞–secure domination number of a graph G.
$\gamma_\infty(G)$	The ∞–order smart domination number of a graph G.

$\gamma^*_\infty(G)$	The ∞–order foolproof domination number of a graph G.
$\mathcal{H}_{p,q}$	A $p \times q$ hexagonal graph.
$i(G)$	The independent domination number of a graph G.
K_n	A complete graph of order n.
$K_{p_1,p_2,\ldots,p_t}$	A complete multipartite graph, with partite set cardinalities $p_1 \leq p_2 \leq \cdots \leq p_t$, $t \in \mathbb{N}$.
$\nu(G)$	The matching number of a graph G.
$N_G(v)$	The open neighbourhood of a vertex v in a graph G.
$N_G[v]$	The closed neighbourhood of a vertex v in a graph G.
$N_G(S)$	The open neighbourhood of a set $S \subseteq V(G)$ in a graph G.
$N_G[S]$	The closed neighbourhood of a set $S \subseteq V(G)$ in a graph G.
$\omega(G)$	The clique number of a graph G.
P_n	A path of order n.
$S_{m\times n}$	A spider consisting of m paths isomorphic to P_n, with one coinciding end–vertex.
$V(G)$	The vertex set of a graph G.
W_n	A wheel of order n.
$w(f)$	The weight of a guard function f.

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Glossary

Acyclic: A *graph* G is called *acyclic* if it does not contain any cycles.

Adjacent: Two *vertices* of a *graph* G are said to be *adjacent* if there exists an *edge* of G *joining* the two *vertices*.

Algorithmic Complexity: *Algorithmic complexity* is a measure of the number of basic operations performed, and the memory expended by an algorithm. If a problem cannot (with current knowledge) be solved by a polynomial time algorithm, it is referred to as an intractable or hard problem, otherwise it is called a tractable problem.

Berge Graph: A *graph* containing neither odd *cycles* of length at least 5, nor their *complements* as *induced subgraphs*, is called a *Berge graph*.

Bipartite: An *n–partite graph* is called *bipartite* if $n = 2$.

Bridge: An *edge* e is called a *bridge* of a *graph* G if the *graph* $G - e$ has more *components* than G.

Cardinality: The number of elements in a set is called its *cardinality*.

Cartesian Product: The *cartesian product* of the *graphs* H_1 and H_2, written as $H_1 \times H_2$, is the *graph* with *vertex set* $V(H_1) \times V(H_2)$, two *vertices* (u_1, u_2) and (v_1, v_2) being *adjacent* in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E(H_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(H_1)$.

Caterpillar: A *tree* is called a *caterpillar* if a *path* results when all the *leaves* are removed.

Clause: A *clause* is a boolean expression involving one or more boolean variables (variables with values 0 or 1) conjoined by means of only the boolean operation OR.

Clique: A *clique* is a *complete subgraph* of a *graph* G that is not an *induced subgraph* of any other *complete subgraph* of G.

Clique Number: The maximum *order* of a *clique* in a *graph* G is called the *clique number* of G, denoted $\omega(G)$.

Clique Partition Number: The minimum number of *cliques* into which a *graph* G may be partitioned is known as the *clique partition number* of G, denoted $\mathfrak{c}(G)$.

Closed Neighbourhood: The *closed neighbourhood* of a *vertex* v in a *graph* G is the set of all *vertices adjacent* to v in G, as well as v itself, and is denoted $N_G[v]$. The *closed neighbourhood* of a *vertex set* S in G is defined as $N_G[S] = \{N_G[v] : v \in S\}$.

Complement: The *complement* $\overline{G}$ of a *graph* G is the *graph* for which $V(\overline{G}) = V(G)$ and $e \in E(\overline{G})$ if and only if $e \notin E(G)$.

Complete Graph: A *complete graph* of *order* n, denoted by K_n, is a *graph* in which every pair of *vertices* are *adjacent.*

Component: A *subgraph* H of a *graph* G is called a *component* of G if H is a maximally *connected subgraph* of G.

Conjunctive Normal Form: A boolean expression is said to be in *conjunctive normal form*, called a cnf–formula, if it comprises several *clauses* conjoined by means of the AND operation.

Connected: For *vertices* u and v of a *graph* G, u is said to be *connected* to v if G contains a $u - v$ *path.* The *graph* G is called a *connected graph* if the *vertices* u and v are connected for any pair $u, v \in V(G)$.

Corona: The *corona* of a *graph* G of order p is the *graph* obtained by joining p new *vertices* to the *vertices* of G by means of a *matching.*

Chromatic Number: A colouring of a *graph* G is an assignment of colours to the *vertices* of G such that no two *adjacent vertices* have the same colour. The minimum number of colours that may be used for such an assignment is called the (vertex) *chromatic number* of G and is denoted $\chi(G)$. If $\chi(G) = n$ for a *graph* G, then the *graph* is said to be n–chromatic.

Cycle: A *cycle* is a *walk* of length $n \geq 3$ in which the begin– and end–vertices, are the same, but in which no other *vertices* repeat. A *graph* consisting of a single *cycle* of length n is so called and denoted C_n.

Degree: The *degree* of a *vertex* v of a *graph* G is the *cardinality* of the *open neighbourhood* of v in G, and is denoted $\deg_G v$.

Deletion: The *deletion* of a non–empty *vertex* subset $S \subseteq V(G)$ from a *graph* G is the *subgraph* with *vertex set* $V(G)\backslash S$ and *edge set* $\{uv \in E(G) : u, v \notin S\}$. Such a *subgraph* is written as $G - S$. For any *edge* subset $J \subseteq E(G)$ the *deletion* of the *edge set* J, written as $G - J$, is the *spanning subgraph* of G with *edge set* $E(G)\backslash J$.

Dominating Set: A *vertex* subset $S \subseteq V(G)$ of G is called a *dominating set* if every *vertex* $v \in V(G)\backslash S$ is *adjacent* to a *vertex* $u \in S$.

Domination Number: The *(lower) domination number*, denoted $\gamma(G)$, of a *graph* G is the minimum *cardinality* over all *minimal dominating sets* of G.

Disconnected: A *graph* that is not *connected* is said to be *disconnected.*

Edge: An *edge* is a 2–element subset of the *vertex set* of a *graph*. *Edges* are indicated by inter–connecting lines between *vertices* in graphical representations of a *graph*.

Edge Set: The set $E(G)$, comprised of all the *edges* of a *graph* G, is called the *edge set* of the *graph*.

Equal: Two *graphs* G and H are said to be *equal*, written as $G = H$, if $V(G) = V(H)$ and $E(G) = E(H)$.

End–vertex: If the *degree* of a *vertex* is 1, then it is called an *end–vertex*.

External Private Neighbourhood: For a *vertex* subset S of a *graph* G, a *vertex* $w \in V(G)\backslash S$ is called an S–external private neighbour (S–epn) of v, if $N(w) \cap S = \{v\}$. The set of all S–epns of v is called the *S–external private neighbourhood* of v, and is denoted $\mathrm{epn}(v, S)$.

Foolproof k^{th}–order ℓ–dominating Function: Let $k, \ell \in \mathbb{N}$. A *foolproof k^{th}–order ℓ–dominating function* ((ℓ, k)–FDF) of a *graph* G is a *safe guard function* $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ of G such that, for any sequence of *vertices* $v_0, v_1, \ldots, v_{k-1}$, moving a *guard* from u_i to v_i results in a *safe guard function* for every $i = 0, 1, \ldots, k-1$, for any sequence of *vertices* $u_i \in N[v_i] \cap (V(G)\backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$.

Foolproof k^{th}–order ℓ–domination Number: The minimum *weight* of an (ℓ, k)–FDF of a *graph* G is denoted $\gamma^*_{\ell,k}(G) = \min_{(\ell,k)\text{–FDFs}}(\sum_{j=1}^{\ell} j|V_j^{(0)}|)$, and is called the *foolproof k^{th}–order ℓ–domination number* of G.

Foolproof k–secure Dominating Function: A *foolproof k–secure dominating function* (k–FSDF) of a *graph* G is a *foolproof k^{th}–order ℓ–dominating function* of G with $\ell = 1$.

Foolproof k–secure Domination Number: The *foolproof k^{th}–order ℓ–domination number* of a *graph* G, in the case where $\ell = 1$, is called the *foolproof k–secure domination number* of G.

Foolproof k–weak Roman Dominating Function: A *foolproof k–weak Roman dominating function* (k–FWRDF) of a *graph* G is a *foolproof k^{th}–order ℓ–dominating function* of G with $\ell = 2$.

Foolproof k–weak Roman Domination Number: The *foolproof k^{th}–order ℓ–domination number* of a *graph* G, in the case where $\ell = 2$, is called the *foolproof k–weak Roman domination number* of G.

Foolproof Secure Dominating Function: A *foolproof secure dominating function* (FSDF) of a *graph* G is a *foolproof k–secure dominating function* of G with $k = 1$.

Foolproof Secure Domination Number: The *foolproof k–secure domination number* of a *graph* G, in the case where $k = 1$, is called the *foolproof secure domination number* of G.

Foolproof Weak Roman Dominating Function: A *foolproof weak Roman dominating function* (FWRDF) of a *graph* G is a *foolproof* k*–weak Roman dominating function* of G with $k = 1$.

Foolproof Weak Roman Domination Number: The *foolproof* k*–weak Roman domination number* of a *graph* G, in the case where $k = 1$, is called the *foolproof weak Roman domination number* of G.

Foolproof ∞–order ℓ–dominating Function: A *foolproof* ∞*–order* ℓ*–dominating function* ((ℓ, ∞)–FDF) of a *graph* G is an (ℓ, k)–FDF of G in the limit as $k \to \infty$.

Foolproof ∞–order ℓ–domination Number: The minimum *weight* of an (ℓ, ∞)–FDF of a *graph* G is denoted $\gamma^*_{\ell,\infty}(G) = \lim_{k\to\infty} \gamma^*_{\ell,k}(G)$, and is called the *foolproof* ∞*–order* ℓ*–domination number* of G.

Foolproof ∞–secure Dominating Function: A *foolproof* ∞*–secure dominating function* (∞–FSDF) of a *graph* G is a *foolproof* ∞*–order* ℓ*–dominating function* of G with $\ell = 1$.

Foolproof ∞–secure Domination Number: The *foolproof* ∞*–order* ℓ*–domination number* of a *graph* G, in the case where $\ell = 1$, is called the *foolproof* ∞*–secure domination number* of G.

Foolproof ∞–weak Roman Dominating Function: A *foolproof* ∞*–weak Roman dominating function* (∞–FWRDF) of a *graph* G is a *foolproof* ∞*–order* ℓ*–dominating function* of G with $\ell = 2$.

Foolproof ∞–weak Roman Domination Number: The *foolproof* ∞*–order* ℓ*–domination number* of a *graph* G, in the case where $\ell = 2$, is called the *foolproof* ∞*–weak Roman domination number* of G.

Forest: A *graph* that is *acyclic*, is called a *forest*, and consists of a number of disconnected *trees*.

Graph: A *graph* is a finite, nonempty set of elements, called *vertices*, together with a (possibly empty) set of 2–element subsets of the *vertex set* called *edges*. A graph may be represented graphically as a set of nodes with inter–connecting lines.

Guard: A *guard* may be seen as a unit of force (or server unit) capable of moving along an *edge* of a *graph*, whose purpose is to *protect* (or service) a *vertex* or set of vertices.

Guard Function: A *guard function* of a *graph* $G = (V, E)$ may be defined as a mapping $f : V \to \mathbb{N}_0$ such that $f(v)$ denotes the number of *guards* stationed at a *vertex* $v \in V$. A *guard function* partitions the *vertex set* V into subsets $V_i = \{v \in V : \ f(v) = i\}$, with $i \in \mathbb{N}_0$. Since there is a one–to–one correspondence between the function f and the ordered partitions $(V_0, V_1, V_2, \ldots)$, a guard function may unambiguously be written as $f = (V_0, V_1, V_2, \ldots)$.

Hexagonal Graph: A *hexagonal graph* $\mathcal{H}_{p,q}$, $p, q \in \mathbb{N}$ is the *union* of the *cartesian product* $P_p \times P_q$, with the *edge sets* $\{v_{2i,j}v_{2i-1,j+1} : \ i = 1, 2, \ldots, \lceil \frac{p}{2} \rceil, \ j = 1, 2, \ldots, q-1\}$ and $\{v_{2i,j-1}v_{2i+1,j} : \ i = 1, 2, \ldots, \lceil \frac{p}{2} \rceil - 1, \ j = 2, 3, \ldots, q\}$.

Incident: A *vertex* v and *edge* e of a *graph* G is said to be *incident*, if e joins v to another *vertex* in G.

Independence Number: The maximum *cardinality* over all *maximal independent sets* of a *graph* G is called the *independence number* of G and is denoted $\beta(G)$.

Independent Domination Number: Any *dominating set* of a *graph* G that is also *independent* is called an independent dominating set of G, the minimum *cardinality* of which is called the *independent domination number*, denoted $i(G)$.

Independent Set: A *vertex* subset S of a *graph* G is called *independent* if no two *vertices* in S are *adjacent* in G.

Induced Subgraph: For a non–empty subset $S \subseteq V(G)$ of a *graph* G the so–called *induced subgraph* of S in G, denoted $\langle S \rangle_G$, is the *subgraph* of G with *vertex set* $V(\langle S \rangle_G) = S$ and *edge set* $E(\langle S \rangle_G) = \{uv \in E(G) : u, v \in S\}$.

Isomorphic: Two *graphs* G and H are called *isomorphic*, written as $G \cong H$, if there exists a one–to–one mapping $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$.

Join: The *join* of two *graphs* H_1 and H_2, written as $H_1 + H_2$, is defined as the *union* of H_1 and H_2 together with all *edges uv* for which $u \in V(H_1)$ and $v \in V(H_2)$. Two *vertices* of a *graph* G are said to be *joined* in G if the the *edge uv* is contained in the *edge set* of G.

k**–Roman Dominating Function:** A *k–Roman dominating function* (kRDF) of a *graph* G is a *safe guard function* $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{k+1}^{(0)})$ of G with the property that, for any sequence of *vertices* $v_0, v_1, \ldots v_{k-1}$, there exists a *vertex* $u_i \in V(G)\backslash V_0^{(i)}$, $i = 0, 1, \ldots, k-1$, in the *neighbourhood* of v_i such that moving a *guard* from u_i to v_i results in a *safe guard function* for every $i = 0, \ldots, k-1$.

k**–Roman Domination Number:** The minimum *weight* of a kRDF of a *graph* G is denoted $\gamma_R^k(G) = \min_{\text{kRDFs}} \sum_{i=1}^{k+1} i|V_i^{(0)}|$, and is called the *$k$–Roman domination number* of G.

Matching: Any 1–*regular subgraph* of a *graph* G is called a *matching* of G. A *matching* of G with the maximum number of *vertices* is called a maximum matching of G.

Maximal Independent Set: An *independent set* S of *vertices* in a *graph* G is called a *maximal independent* set if S is not a proper subset of any other *independent set* of G.

Minimal Dominating Set: A *dominating set* S of a *graph* G is called a *minimal dominating* set if no proper subset of S is a *dominating set* of G.

Multipartite: An *n–partite graph* is called *multipartite* if $n > 2$.

n**–partite:** A *graph* G is called n–partite, $n \geq 2$, if the vertex set may be partitioned into n subsets, such that no *edge* of G connects *vertices* from the same subset.

Open Neighbourhood: The *open neighbourhood* of a *vertex* v in a *graph* G is the set of all *vertices adjacent* to v in G, and is denoted $N(v)$. The *open neighbourhood* of a set S is defined as $N[S] = \{N[v] : \ v \in S\}$.

Order: The *cardinality* of the *vertex set* of a *graph* G is called the *order* of G.

Packing: A set $S \subseteq V(G)$ is called a *packing* in G if $N[u] \cap N[v] = \emptyset$ for every pair $u, v \in S$ (in other words, the shortest *path* between any pair of *vertices* in S is at least 3 in G).

Path: A *walk* in which no *vertex* is repeated is called a *path*. A *graph* solely consisting of a *path* of order n is so called and denoted P_n.

Perfect Graph: A *graph* G is called a *perfect graph* if $\omega(\langle S\rangle_G) = \chi(\langle S\rangle_G)$ for all $S \subseteq V(G)$, and $\beta(\langle S\rangle_G) = \mathfrak{c}(\langle S\rangle_G)$ for all $S \subseteq V(G)$.

Perfect Matching: A *perfect matching* of a *graph* G, if it exists, is a *matching* of G containing all the *vertices* of G.

Protect: For a *safe guard function* f, if the movement of a *guard* from an occupied *vertex* u to a *vertex* v, results in a *safe guard function*, the *vertex* u is said to *protect* v under f.

Regular: A *graph* G is called r*–regular* if each *vertex* of G has *degree* r. A *graph* is referred to as *regular* if it is r*–regular* for some $r \in \mathbb{N}_0$.

Roman Dominating Function: A *Roman dominating function* (RDF) of a *graph* G is a *safe guard function* $f = (V_0, V_1, V_2)$ of G satisfying the condition that every *vertex* $v \in V_0$ is adjacent to at least one *vertex* $u \in V_2$.

Roman Domination Number: The minimum *weight* of an RDF of a *graph* G is denoted $\gamma_R(G) = \min_{\text{RDFs}}(|V_1| + 2|V_2|)$ and is called the *Roman domination number* of G.

Safe Guard Function: A *guard function* $f = (V_0, V_1, \ldots)$ of a *graph* G is called a *safe guard function* of G if each *vertex* $v \in V_0$ is *adjacent* to some *vertex* $u \in V(G)\backslash V_0$.

Secure Dominating Function: See *smart secure dominating function.*

Secure Domination Number: See *smart secure domination number.*

Size: The *cardinality* of the *edge set* of a *graph* G is called the *size* of G.

Smart k^{th}–order ℓ–dominating Function: Let $k, \ell \in \mathbb{N}$. A *smart* k^{th}*–order* ℓ*–dominating function* ((ℓ, k)–SDF*)* of a *graph* G is a *safe guard function* $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ of G, with the property that for any sequence of *vertices* $v_0, v_1, \ldots, v_{k-1}$, there exists a sequence of *vertices* $u_i \in N[v_i] \cap (V(G)\backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$, such that moving a *guard* from u_i to v_i results in a *safe guard function* for every $i = 0, 1, \ldots, k-1$.

Smart k^{th}–order ℓ–domination Number: The minimum *weight* of an (ℓ,k)–SDF of a *graph* G is denoted $\gamma_{\ell,k}(G) = \min_{(\ell,k)-\text{SDFs}}(\sum_{j=1}^{\ell} j|V_j^{(0)}|)$, and is called the *smart k^{th}–order ℓ–domination number* of G.

Smart k–secure Dominating Function: A *smart k–secure dominating function* (k–SSDF) of a *graph* G is a *smart k^{th}–order ℓ–dominating function* of G with $\ell = 1$.

Smart k–secure Domination Number: The *smart k^{th}–order ℓ–domination number* of a *graph* G, in the case where $\ell = 1$, is called the *smart k–secure domination number* of G.

Smart k–weak Roman Dominating Function: A *smart k–weak Roman dominating function* (k–SWRDF) of a *graph* G is a *smart k^{th}–order ℓ–dominating function* of G with $\ell = 2$.

Smart k–weak Roman Domination Number: The *smart k^{th}–order ℓ–domination number* of a *graph* G, in the case where $\ell = 2$, is called the *smart k–weak Roman domination number* of G.

Smart Secure Dominating Function: A *smart secure dominating function* (SSDF) of a *graph* G is a *smart k–secure dominating function* of G with $k = 1$.

Smart Secure Domination Number: The *smart k–secure domination number* of a *graph* G, in the case where $k = 1$, is called the *smart secure domination number* of G.

Smart Weak Roman Dominating Function: A *smart weak Roman dominating function* (SWRDF) of a *graph* G is a *smart k–weak Roman dominating function* of G with $k = 1$.

Smart Weak Roman Domination Number: The *smart k–weak Roman domination number* of a *graph* G, in the case where $k = 1$, is called the *smart weak Roman domination number* of G.

Smart ∞–order ℓ–dominating Function: A *smart ∞–order ℓ–dominating function* ((ℓ,∞)–SDF) of a *graph* G is an (ℓ,k)–FDF of G in the limit as $k \to \infty$.

Smart ∞–order ℓ–domination Number: The minimum *weight* of an (ℓ,∞)–SDF of a *graph* G is denoted $\gamma^*_{\ell,\infty}(G) = \lim_{k\to\infty} \gamma^*_{\ell,k}(G)$, and is called the *smart ∞–order ℓ–domination number* of G.

Smart ∞–secure Dominating Function: A *smart ∞–secure dominating function* (∞–SSDF) of a *graph* G is a *smart ∞–order ℓ–dominating function* of G with $\ell = 1$.

Smart ∞–secure Domination Number: The *smart ∞–order ℓ–domination number* of a *graph* G, in the case where $\ell = 1$, is called the *smart ∞–secure domination number* of G.

Smart ∞–weak Roman Dominating Function: A *smart ∞–weak Roman dominating function* (∞–SWRDF) of a *graph* G is a *smart ∞–order ℓ–dominating function* of G with $\ell = 2$.

Smart ∞–weak Roman Domination Number: The *smart ∞–order ℓ–domination number* of a *graph* G, in the case where $\ell = 2$, is called the *smart ∞–weak Roman domination number* of G.

Spanning Subgraph: A *graph* H is called a *spanning subgraph* of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

Spider: A *spider* is a *graph* consisting of a number of equally sized *paths* with one coinciding *end–vertex*.

Star: The *bipartite graph* $K_{1,n} \cong K_{n,1}$ is a often called an *n–star*, $n \in \mathbb{N}$.

Subgraph: A *graph* H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Support Vertex: Any *vertex adjacent* to a *leaf* of a *graph* G is called a *support vertex* of G, while an *r–support vertex* of G is a *vertex adjacent* to at least r *leaves* of G.

Tree: A *tree* is an *acyclic connected graph.*

Union: The *union* of two *graphs* H_1 and H_2, written as $H_1 \cup H_2$, is the *graph* H with *vertex set* $V(H) = V(H_1) \cup V(H_2)$ and *edge set* $E(H) = E(H_1) \cup E(H_2)$.

Upper Domination Number: The maximum *cardinality* over all *minimal dominating sets* of a *graph* G is called the *upper domination number* of G, denoted $\Gamma(G)$.

Vertex: A *vertex* is a combinatorial element in terms of which a *graph* is defined. *Vertices* are indicated by nodes in the graphical representation of a *graph.*

Vertex Set: The set comprised of all *vertices* of a *graph* G, is called the *vertex set* of G.

Walk: A *walk* in a *graph* G is an alternating sequence of *incident vertices* and *edges.* The number of *edges* in the *walk* defines its length, while the number of *vertices* defines its order.

Weak Roman Dominating Function: See *smart weak Roman dominating function.*

Weak Roman Domination Number: See *smart weak Roman domination number.*

Weight: The *weight* of a *guard function* f is the total number of *guards* deployed under f and is denoted $w(f) = \sum_{v \in V} f(v)$.

Wheel: A *wheel* W_n of order n may be defined as the *join* of a *cycle* of order n with another *vertex*, sometimes referred to as the hub of the *wheel.*

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Chapter 1

Introduction

1.1 Historical Background

During its domination of Europe in the third century A.D., the Roman empire had 50 legions at its command (each consisting of various infantry and cavalry units, [20]), to deploy and secure even the farthest reaches of its territories. Losing much of its power, however, the empire had only 25 legions available by the following century. Emperor Constantine the Great (274–337 A.D.) faced the problem of efficiently deploying the limited number of legions at his disposal, while attempting to protect the entire empire. A grouping of six legions, called a field army, was deemed sufficient to secure any one region of the empire. Thus four complete field armies were available to the Emperor. Considering a simplification of the geographical area, there were eight regions where field armies could be stationed, as illustrated by the map in Figure 1.1.

Pectora roborant cultus recti

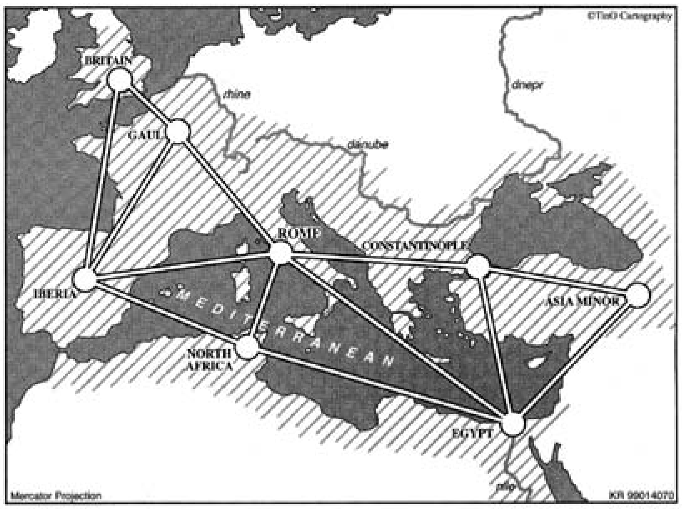


Figure 1.1: *The various regions of the Roman empire during the 3rd and 4th century A.D. (Reproduced from [23].)*

A deployment would secure the entire mapped area if every region was either occupied by a field army, or if it was directly adjacent to a region that was occupied by two field armies. The emperor decreed that two field armies be stationed at a region before one would be allowed to move to an unoccupied, neighbouring region, in an attempt to ensure that the region vacated by the moving field army could not be successfully attacked by an enemy. It is reasonable to expect that the limited number of legions at his disposal caused the emperor to be torn between his political strategies and the following two questions:

(a) What is the minimum number of field armies needed to secure the empire?

(b) If the available number is less than this minimum, how should the field armies be stationed in order to defend the largest number of regions?

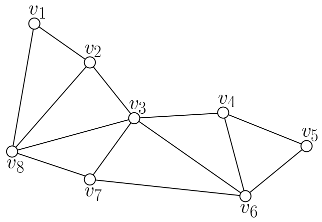
In order to answer these two questions, the problem of Constantine the Great is cast in a more general setting in the next section.

1.2 Problem Description and History

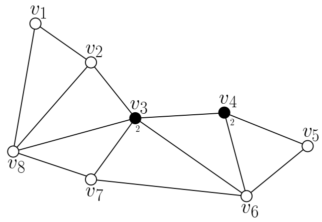
Emperor Constantine's problem of successfully placing field armies throughout the Roman empire, as discussed in §1.1, may well be the first recorded location problem. To maintain generality in the informal problem description of this section, the Roman field armies will be referred to as *guards*. Presently, many practical situations occur in which it is neccessary to deploy a number of *guards* (or resources) so as to *secure* (supply) some given *area* (facility with a certain service). In such cases it is usually beneficial to minimise the number of guards required, while still securing or serving the entire area. Considering the problem of securing the Roman empire, the mapped area in Figure 1.1 may be modelled by a graph, which may be seen informally[1] as a set of nodes on a two–dimensional plane, as well as inter–connecting lines between these nodes, denoting adjacency of regions, as shown in Figure 1.2(a). In order to secure the entire region modelled by the graph, the set of occupied nodes (i.e. nodes at which at least one guard is stationed) has to, at the very least, form a so–called dominating set for the graph. A set of occupied nodes is said to form a **dominating set** if each unoccupied node (i.e. a node at which no guards are stationed) is directly adjacent (joined by means of a line) to some occupied node. The reader is referred to Appendix A.1 for additional practical motivation, as mentioned in [25].

A safe guard function may be defined informally as a deployment of guards on a graph of nodes and inter–connecting lines, such that the set of occupied nodes forms a dominating set of the graph, in which case the number of guards deployed throughout the graph is called the **weight** of the safe guard function. For a given graph, G say, the minimum attainable weight of a safe guard function for that graph, is called the **domination number** of the graph, denoted by $\gamma(G)$. It is easily verified that the domination number of the graph G_1 in Figure 1.2(a) (which models the area in Figure 1.1) is $\gamma(G_1) = 2$. For example, if the nodes v_6 and v_8 each receive one guard, a safe guard function of weight 2 is achieved. It is clearly impossible for one guard to dominate the graph, irrespective of its deployment. So the Roman empire required at the very least two field armies to secure its region shown in Figure 1.1.

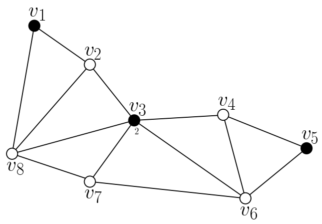
[1]The notion of a graph, its properties and the description of the general problem (of which Constantine's defence problem forms a special case) will be made more precise (in a mathematical sense) in the following chapter.



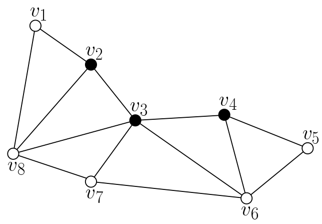
(a) *The graph G_1 of nodes and inter–connecting lines used to model the geographical area of the Roman empire*



(b) *The deployment strategy decided upon by Emperor Constantine.*



(c) *An example of a minimum weight Roman dominating function of the graph G_1.*



(d) *An example of a minimum weight weak Roman dominating function of the graph G_1.*

Figure 1.2: *(a) The graph G_1 of nodes and inter–connecting lines used to model the geographical area of the Roman empire during the 3rd and 4th century A.D. The various regions are: $v_1 \equiv$ Britain, $v_2 \equiv$ Gaul, $v_3 \equiv$ Rome, $v_4 \equiv$ Constantinople, $v_5 \equiv$ Asia Minor, $v_6 \equiv$ Egypt, $v_7 \equiv$ North Africa, $v_8 \equiv$ Iberia. (b) The deployment strategy decided upon by Emperor Constantine. Occupied vertices are indicated as dark vertices, while vertices with two guards stationed at them are so indicated. (c) An example of a minimum weight Roman dominating function of the graph G_1. (d) An example of a minimum weight weak Roman dominating function of the graph G_1.*

In addition to requiring the deployment of guards to form a dominating set of the area, Emperor Constantine decreed further restrictions for securing the Roman empire, as described in §1.1. Prompted by these additional restrictions, as well as the papers by Revelle and Rosing [25] and Stewart [27], Cockayne *et al.* [6] established the notion of so–called Roman domination of a graph. This concept is more restrictive than the above mentioned classical notion of domination. A **Roman dominating function** is defined as a safe guard function, with the added condition that any unoccupied node is directly adjacent to an occupied node with two guards stationed at it. This requirement conforms to the discussion in §1.1. For a given graph G, the minimum number of guards in a deployment forming a Roman dominating function, is called the **Roman domination number** of the graph, and denoted by $\gamma_R(G)$. Emperor Constantine decided to compromise the defense of Britain by placing two field armies in Rome and two at his new capital Constantinople, a deployment shown in Figure 1.2(b). It is, in fact, now known that the Roman domination number for the graph G_1 in Figure 1.2(a) is $\gamma_R(G_1) = 4$, as calculated in [25], using an integer programming technique. This shows that at least four guards (field armies in this case) was needed to secure the Roman empire, as shown in Figure 1.1. Stationing one guard at each of the nodes v_1 and v_5, and two guards at v_3, provides an example of a Roman dominating function of minimum weight for the empire, as illustrated in Figure 1.2(c), since any unoccupied node is directly adjacent to a node with two guards stationed at it. It may easily be verified by way of trial and error that a Roman dominating function of weight 3 does not exist for the graph in Figure 1.2(a), since any deployment of 3 guards will necessarily leave at least one unoccupied node not directly adjacent to a node with two guards stationed at it. It is concluded that Emperor Constantine's decision to compromise the defense of Britain was not absolutely neccessary, although other factors probably played a role in his decision.

If it is assumed that no two nodes will be attacked simultaneously (a possibly dangerous assumption), then it might be possible for the emperor to save significantly on the number of guards required to defend the empire. These resources saved may be used to strengthen the defenses of other vital locations. With this in mind, Henning & Hedetniemi [17] suggested relaxing the definition of Roman domination to arrive at the notion of so–called weak Roman domination. A **weak Roman dominating function** still requires maximally two guards stationed at a node, but any unoccupied node need only be directly adjacent to some occupied node, having either one or two guards stationed at it. Additionally, it is required that for any unoccupied node there exists a directly adjacent, occupied node, such that moving a guard from that node to the unoccupied node, again results in the deployment being a safe guard function. The **weak Roman domination number** for a graph G, denoted by $\gamma_r(G)$, is the minimum number of guards needed to form a weak Roman dominating function when deployed. Since this value for the graph G_1 in Figure 1.2(a) is $\gamma_r(G_1) = 3$, it is known that a minimum of three guards would have been needed to secure the empire against a single attack. A possible deployment of such a weak Roman dominating function of minimum weight is achieved by stationing one guard at each of the nodes v_2, v_3 and v_4, as illustrated in Figure 1.2(d). With this deployment, it may be verified that, for any unoccupied node, at least one adjacent guard exists such that moving that guard to the node in question, results in a safe guard function. No deployment of 2 guards will achieve these requirements, which verifies that the weak Roman domination number is indeed equal to 3.

Observing that the guards of a minimum weight weak Roman dominating function may be deployed with maximally one guard per node in the case of the Roman empire, the definition of weak Roman domination was broadened yet further by Cockayne *et al.* [8] to the notion of secure domination. The concept of a **secure dominating function** is similar to that of a weak Roman dominating function, with the exception that the number of guards stationed at a node is limited to at most one. For a given graph G, the minimum number of guards needed to form a secure dominating function is called the **secure domination number**, $\gamma_s(G)$, of the graph. For the graph G_1 in Figure 1.2(a), the secure domination number is $\gamma_s(G_1) = 3$; a deployment of guards at the nodes v_2, v_3 and v_4 being an example of a minimum weight secure dominating function.

Burger *et al.* [2] noted a significant difference between the definition of domination and Roman domination on the one hand, and weak Roman domination and secure domination on the other. The difference lies in the former two notions being static in nature, in the sense that no guard movements are considered, whereas the latter two notions possess a dynamic characteristic. This aspect of a dynamic guard configuration results from requiring domination of a graph both before and after the movement of a guard from an occupied to an unoccupied node. It is this dynamic domination characteristic that led to the notion of higher order domination, as initially explored by Burger *et al.* [2] and Henning [18] independently.

In formalising the definition of higher order domination, it was acknowledged that the notion of dynamic domination does not have to be limited to just one move, but may involve any prespecified number of moves, even allowing infinitely many moves in a bid to render the graph perpetually secure. However, the restriction of maximally two guards per node may also be alleviated to allow any prespecified maximal number of guards per node. Two further distinctions were made by Burger *et al.* [2, 3]: Protection or defense strategies for graphs may simply require the existence of a guard–move to a node resulting in a safe guard function, leaving it up to the strategist to decide on the movement strategy. Such strategies may be referred to as **smart domination** strategies. On the other hand, protection strategies may be required to be so robust as to allow for any guard–move from an occupied node resulting in a safe guard function. Such strategies may be referred to as **foolproof domination** strategies.

The hierarchial structure in Figure 1.3 shows the various types of higher order domination parameters established by Burger *et al.* Each parameter may be classified as either being in the category of smart or foolproof domination. Thereafter, it depends on the maximum number of guards allowed at a node, as well as whether a finite or infinite number of moves must be catered for. Considering Figure 1.3, it is noted that the smart domination number $\gamma_{1,0}(G)$ for a finite number of 0 moves on a graph G with maximally one guard per vertex, is in fact equivalent to the classical domination number, $\gamma(G)$. Also, the parameter $\gamma_{1,1}(G)$ for a graph G is noted to be the secure domination parameter, $\gamma_s(G)$, as introduced above. Furthermore, the smart domination number $\gamma_{2,1}(G)$ for the finite number of 1 move on a graph G with maximally 2 guards per vertex, is noted to be the weak Roman domination number $\gamma_r(G)$. If an additional restriction is introduced, requiring each unoccupied vertex to be directly adjacent to a vertex with two guards stationed at it, then the parameter $\gamma_{2,0}(G)$ is merely the Roman domination number $\gamma_R(G)$. As is evident from Figure 1.3, various other higher order domination parameters exist, most of which are (to the knowledge of the author) still unexplored.

1.3 Thesis Objectives

As discussed in the previous section, the parameters $\gamma_{\ell,k}$ and $\gamma^*_{\ell,k}$ (referring to Figure 1.3) were introduced by Burger *et al.* [2, 3], for the case where $\ell \in \{1, 2\}$, also catering for the possibility of k being infinitely large. Furthermore, Henning [18] considered the smart finite order parameter $\gamma_{k+1,k}$. The parameters studied in this thesis are generalised to allow for an arbitrary value of maximally ℓ, say, guards stationed at each node. It is expected that most of the properties obtained in [2, 3, 18] hold for these generalised parameters. This leads to the first objective of this thesis.

Objective I: To introduce a framework for the above mentioned generalised higher order domination, to put forth a comprehensive survey of the known results on this topic (mainly from [2, 3, 18]), and to examine and compare these results in this, more general, framework, so as to create a reference work summarising the state of the art.

As a consequence of this objective, the main body of the thesis contains a number of results originally obtained in [2, 3, 18], among others, modified only slightly to acommodate the generalisation.

As the work by Burger *et al.* [2, 3] and Henning [18] was the first to investigate the protection of a graph against an arbitrary number of k attacks, the higher order domination parameters are still relatively unexplored. A second objective of this thesis is to obtain, where possible, additional results pertaining to general graphs, as well as some special graph classes.

Objective II: To obtain additional properties on the higher order domination parameters, to investigate known general bounds, to establish parameter values for some special graph classes, and to provide a clear indication of unresolved problems regarding the parameters.

1.4 Thesis Overview

This thesis consists of six chapters, in addition to the present introductory chapter.

Chapter 2 provides an overview of basic graph and complexity theoretic concepts used throughout the rest of the thesis. Chapter 3 comprises a review of the combinatorial literature on Roman domination, weak Roman domination, secure domination and higher order domination. These notions of graph domination are defined formally in terms of the unifying notation used in the following chapters, and results achieved on the various domination numbers are discussed.

Chapters 4–6 constitute the main body of the thesis. Chapter 4 opens with a formal graph theoretic definition of the notion of higher order domination, as initially introduced by Burger *et al.* [2]. Domination numbers depending on a finite number of guard–moves that are required to result in a safe guard function, are explored. Some of the known results from [2] and [18] are discussed and generalised where possible, while various new results on finite order domination numbers are established.

Higher order domination numbers, catering for the possibility of graph protection against an infinite number of attacks, by way of perpetual guard movements resulting in safe guard functions (as introduced by Burger *et al.* [3]) are considered in Chapter 5. Results on these infinite order domination numbers, additional to the results in [3], are established.

In Chapter 6, the various higher order domination numbers are studied in the contexts of various well–known graph classes. By focussing on a specific class of graphs, certain characteristics of the graph structure may be exploited, enabling one to obtain much sharper results for the different classes of domination numbers.

The thesis is concluded in Chapter 7 with a summary of the results achieved. Other, novel variants of higher order domination are proposed and areas requiring further exploration are suggested.

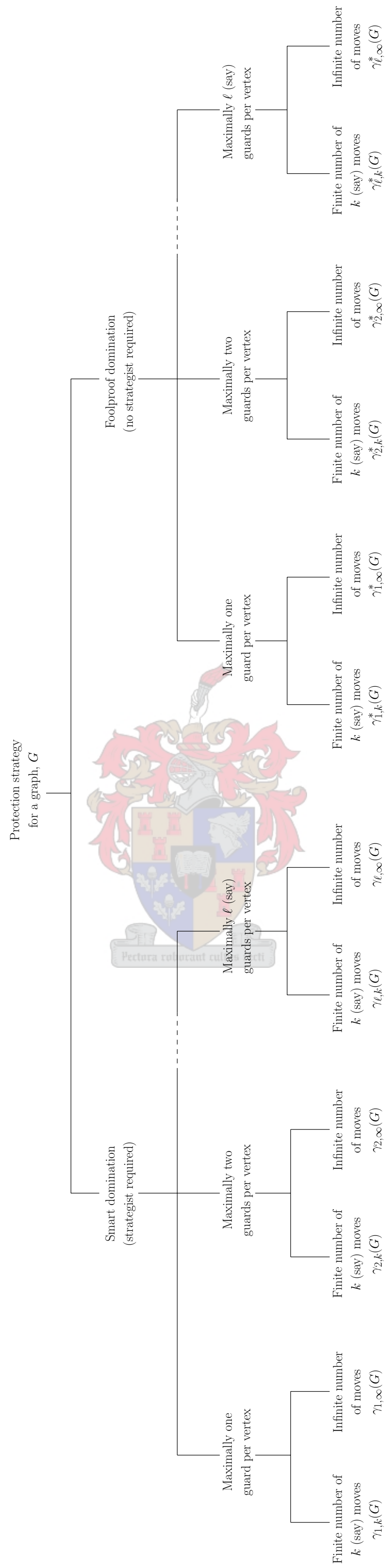


Figure 1.3: *Hierarchial structure of the various types of higher order domination parameters.*

Chapter 2

Basic Concepts in Graph and Complexity Theory

This chapter introduces the graph theoretic definitions required for this thesis in §2.1, as well as an overview of basic complexity theoretic concepts in §2.2.

2.1 Basic Graph Theoretic Concepts

A **graph** $G = (V, E)$ is a finite, nonempty set $V(G)$, together with a (possibly empty) set $E(G)$ of 2–element subsets of $V(G)$. The elements of V are called **vertices**, while those of E are called **edges**. The number of vertices in a graph G is called the **order** of G, denoted by $p = |V(G)|$, while the number of edges in G is called the **size** of G, denoted by $q = |E(G)|$. A graph of order p and size q is often referred to as a (p, q)–graph. If the unordered pair $e = \{u, v\}$ is an edge of the graph G, informally written as $e = uv$, it is said that the vertices u and v are **adjacent** in G and that the edge e joins u and v. The edge e is said to be incident with the vertices u and v. A graphical representation of an order 7 graph G_1 of size 8 is shown in Figure 2.1. The vertex set is $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and the edge set is $E(G_1) = \{v_1v_6, v_1v_7, v_2v_4, v_3v_5, v_3v_6, v_3v_7, v_4v_5, v_5v_6\}$. The vertices v_1 and v_6 are adjacent in G_1, while v_1 and v_2 are not.

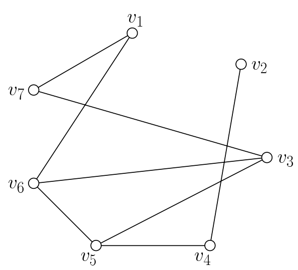


Figure 2.1: *Graphical representation of a (7,8)–graph,* G_1.

2.1.1 Neighbourhoods

The **open neighbourhood** of a vertex v in a graph G is defined as the set

$$N_G(v) = \{u \in V(G) : \ uv \in E(G)\},$$

while the **closed neighbourhood** of v in G is defined as

$$N_G[v] = N_G(v) \cup \{v\}.$$

The open neighbourhood of a set S is defined as $N(S) = \{N(v) : \ v \in S\}$, while the closed neighbourhood of a set S is defined as $N[S] = \{N[v] : \ v \in S\}$. For any vertex v in a graph G, the number of vertices adjacent to v, i.e. $|N_G(v)|$, is called the **degree** of v in G, denoted by $\deg_G v$. Note that if the reference to a graph G is clear from the context, the subscript is often omitted, hence written as $\deg\ v$ only. If the degree of a vertex is 0, it is called an isolated vertex, while if the degree is 1, it is called an end–vertex. The minimum degree of vertices in G is denoted by $\delta(G)$, while the maximum degree of the vertices is denoted by $\Delta(G)$. Referring to the graph G_1 in Figure 2.1, the open neighbourhood of the vertex v_5 is $N_{G_1}(v_5) = \{v_3, v_4, v_6\}$, while its closed neighbourhood is $N_{G_1}[v_5] = \{v_3, v_4, v_5, v_6\}$. The graph has no isolated vertices, but v_2 is, in fact, an end–vertex. The minimum degree of G_1 is therefore $\delta(G_1) = 1$, while the maximum degree is $\Delta(G_1) = 3$.

The following theorem, often referred to as the **Fundamental Theorem of Graph Theory**, is probably one of the most well–known results in the discipline and relates the sum total of the degrees and the size of any graph.

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Theorem 2.1 *Let G be a (p, q)–graph, with $V(G) = \{v_1, v_2, \ldots, v_p\}$. Then*

$$\sum_{i=1}^{p} \deg_G v_i = 2q.$$

Proof: When the degrees of all the vertices are summed, each edge is counted twice, once for each of the vertices that it joins. ■

For a vertex subset S of a graph G, a vertex $w \in V(G)\backslash S$ is called an S**–external private neighbour** (S–epn) of v, if $N(w) \cap S = \{v\}$. The set of all S–epn's of v is denoted by $\text{epn}(v, S)$. Considering the vertex subset $S = \{v_5, v_6\}$ of G_1, the vertex v_4 is an S–epn of v_5, while v_3 is not an external private neighbour of any vertex in S. A vertex subset $S \subseteq V(G)$ of a graph G is called an irredundant set of G if, for every vertex $v \in S$, $\text{epn}(v, S) \neq \emptyset$ or v is an isolated vertex in $\langle S \rangle_G$. In other words, S is irredundant if every vertex in S has at least one external private neighbour, or is not adjacent to any other vertex in S. Again considering the graph G_1 of Figure 2.1, the set $S = \{v_5, v_6\}$ is irredundant since v_5 has v_4 as an S–epn and v_6 has v_1 as an S–epn. For the purposes of this thesis, an external private neighbour will simply be referred to as a private neighbour.

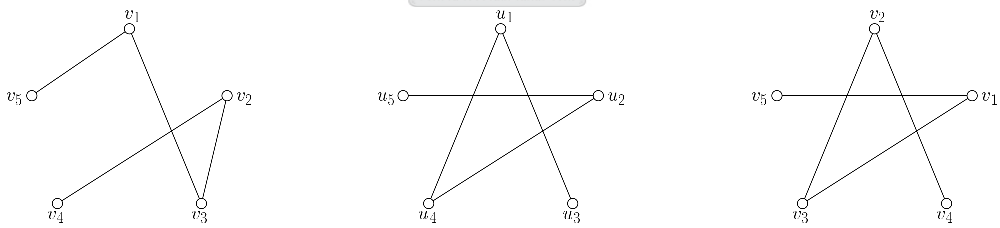
(a) G_2 (b) $\overline{G}_2$

Figure 2.2: *Illustration of a graph G_2 and its complement.*

2.1.2 Graph Complements, Isomorphisms and Subgraphs

The **complement** $\overline{G}$ of a graph G is the graph for which $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A $(5,4)$–graph G_2 is shown in Figure 2.2(a), while its complement $\overline{G}_2$ is the $(5,6)$–graph shown in Figure 2.2(b).

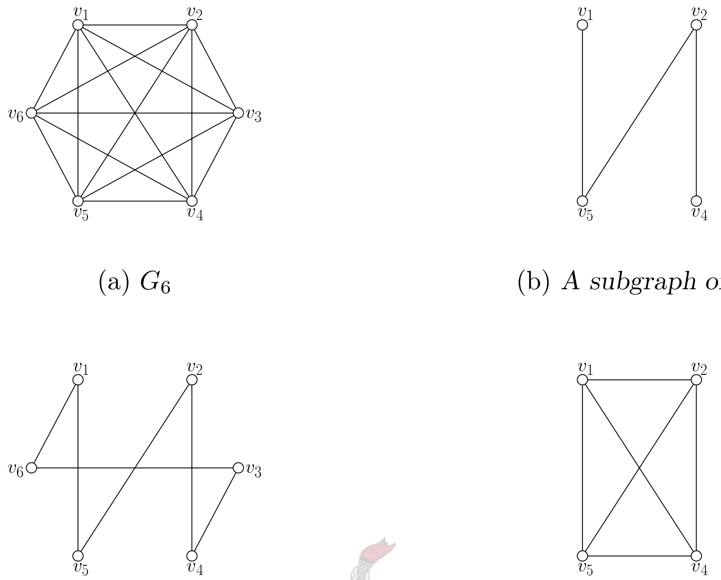
Two graphs G and H are called **isomorphic**, written as $G \cong H$, if there exists a one–to–one mapping $\phi : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. The function ϕ is called an isomorphism. If ϕ maps G onto itself, it is called an automorphism. Two graphs G and H are said to be **equal** if $V(G) = V(H)$ and $E(G) = E(H)$. Therefore, equal graphs are isomorphic, but the converse is not true. The graph G_4 shown in Figure 2.3(b) is isomorphic (but not equal) to G_3, shown in Figure 2.3(a), while G_5, shown in Figure 2.3(c), is both equal and isomorphic to G_3.



(a) *The graph G_3.* (b) *The graph G_4 is isomorphic to G_3.* (c) *The graph G_5 is equal to G_3.*

Figure 2.3: *Illustration of isomorphism and equality in graphs.*

A graph H is called a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and is called a **spanning subgraph** of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. For a non–empty vertex subset $S \subseteq V(G)$ of a graph G the so–called **induced subgraph** of S in G, denoted by $\langle S \rangle_G$, is the subgraph of G with vertex set $V(\langle S \rangle_G) = S$ and edge set $E(\langle S \rangle_G) = \{uv \in E(G) : u, v \in S\}$. The graph shown in Figure 2.4(b) is an example of a subgraph of G_6, shown in Figure 2.4(a), while the graph in Figure 2.4(c) is a spanning subgraph of G_6. Lastly, the induced subgraph $\langle\{v_1, v_2, v_4, v_5\}\rangle_{G_6}$ is illustrated in Figure 2.4(d). For a given graph F, a graph G is called F–**free** if G does not contain an induced subgraph isomorphic to F. If $F \cong K_{1,3}$, an F–free graph is often called claw–free.

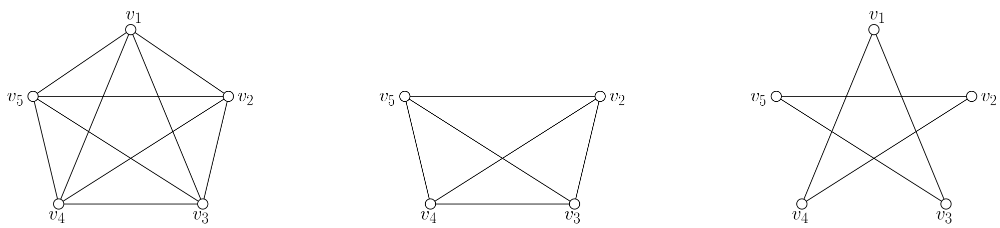


(a) G_6 (b) *A subgraph of* G_6

(c) *A spanning subgraph of* G_6 (d) *An induced subgraph* $\langle\{v_1, v_2, v_4, v_5\}\rangle$ *of* G_6

Figure 2.4: *Illustration of a subgraph, spanning subgraph and induced subgraph of the graph* G_6*.*

The **deletion** of a non–empty vertex subset $S \subseteq V(G)$ from a graph G is the subgraph with vertex set $V(G)\backslash S$ and edge set $\{uv \in E(G) : \ u, v \notin S\}$. Such a subgraph is denoted by $G - S$. For any edge subset $J \subseteq E(G)$ the deletion of the edge set J, denoted by $G - J$, is the spanning subgraph of G with edge set $E(G)\backslash J$. Considering the graph G_7 in Figure 2.5(a), with vertex subset $S = \{v_1\}$ and edge subset $J = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$, the subgraph $G_7 - S$ is shown in Figure 2.5(b), while $G_7 - J$ is shown in Figure 2.5(c).



(a) G_7 (b) $G_7 - S$, *for* $S = \{v_1\}$ (c) $G_7 - J$, *for* $J = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$

Figure 2.5: *Illustration of the deletion of a vertex and edge subset respectively.*

2.1.3 Connectedness

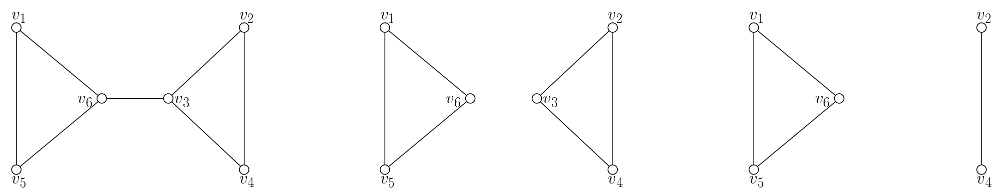
A **walk** in a graph G is an alternating sequence of vertices and edges

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{i-1}, e_i, v_i, \ldots, v_{n-1}, e_n, v_n,$$

also called a $v_0 - v_n$ walk, such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, n$. The number of edges in the walk defines its length, while the number of vertices defines its order. When referring to a walk, the edges are often omitted where ambiguity is impossible. An example of a walk in the graph G_7 in Figure 2.5(a) is v_1, v_3, v_5, v_1, v_4. A walk in which no edge is repeated is called a trail, while a walk in which no vertex is repeated is called a **path**. A **cycle** is a walk of length $n \geq 3$ in which the begin– and end–vertices, v_0 and v_n, are the same, but in which no other vertices repeat. Considering the graph G_7 in Figure 2.5(a), the walk v_1, v_3, v_5 is a path of order 3 and length 2, while v_1, v_3, v_5, v_1 is a cycle of length 3. Furthermore, a set $S \subseteq V(G)$ is called a **packing** in G if $N[u] \cap N[v] = \emptyset$ for every pair $u, v \in S$ (in other words, the shortest path between any pair of vertices in S is at least 3).

For vertices u and v of a graph G, u is said to be connected to v if G contains a $u - v$ path. The graph G is called a **connected** graph if the vertices u and v are connected for any pair $u, v \in V(G)$. A graph that is not connected is said to be **disconnected**. A subgraph H of G is called a **component** of G if H is a maximally connected subgraph of G. An edge e is called a **bridge** of G if the graph $G - e$ has more components than G, and a vertex v is called a **cut–vertex** of G if the graph $G - v$ has more components than G. Therefore, an edge e in a connected graph G is a bridge if $G - e$ is disconnected and a vertex v in a connected graph G is a cut–vertex if $G - v$ is disconnected. The graph G_8 shown in Figure 2.6(a) has the edge v_3v_6 as a bridge, while v_3 is a cut–vertex of G_8. The following theorem gives a characterisation of when an edge of a graph is a bridge. A proof of this theorem may be found in [5], pp. 22-23.

Theorem 2.2 *An edge e of a connected graph G is a bridge of G if and only if e does not lie on a cycle of G.* ■



(a) G_8 (b) $G_8 - \{v_3v_6\}$ (c) $G_8 - v_3$

Figure 2.6: *Illustration of a bridge and cut–vertex in the connected graph G_8 in (a). (b) The edge v_3v_6 is a bridge, since $G_8 - \{v_3v_6\}$ is disconnected. (c) The vertex v_3 is a cut–vertex, since $G_8 - v_3$ is disconnected.*

2.1.4 Graph Unions, Joins and Products

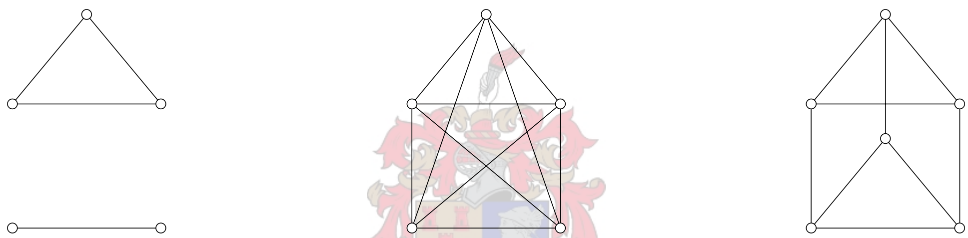
Graphs may be produced from other graphs in several ways. The **union** of two graphs H_1 and H_2, denoted by $H_1 \cup H_2$, is the graph H with vertex set $V(H) = V(H_1) \cup V(H_2)$ and edge set $E(H) = E(H_1) \cup E(H_2)$. The **join** of two graphs is denoted by $H_1 + H_2$ and is the union of H_1 and H_2 as well as all edges uv with $u \in V(H_1)$ and $v \in V(H_2)$. The **cartesian product** of the graphs H_1 and H_2, denoted by $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices (u_1, u_2) and (v_1, v_2) being adjacent in $H_1 \times H_2$ if and only if either

$$u_1 = v_1 \text{ and } u_2 v_2 \in E(H_2),$$

or

$$u_2 = v_2 \text{ and } u_1 v_1 \in E(H_1).$$

From the symmetry in the definition it follows that $H_1 \cup H_2 \cong H_2 \cup H_1$, $H_1 + H_2 \cong H_2 + H_1$ and $H_1 \times H_2 \cong H_2 \times H_1$. These concepts are illustrated in Figure 2.7(a)–(c) for the graphs C_3 and P_2.



(a) $C_3 \cup P_2$ (b) $C_3 + P_2$ (c) $C_3 \times P_2$

Figure 2.7: *Illustration of (a) the union of the graphs C_3 and P_2, (b) the join of the graphs C_3 and P_2 and (c) the cartesian product of the graphs C_3 and P_2.*

2.1.5 Special Graphs

A graph solely consisting of a **path** of order n is so called and denoted by P_n. Similarly, a graph consisting of a single **cycle** of length n is so called and denoted by C_n. Paths and cycles are called odd [or even] if they have odd [or even] lengths.

A graph G is called r–**regular** if each vertex of G has degree r. A graph is referred to as regular if it is r–regular for some $r \in \mathbb{N}_0$. Any 1–regular subgraph of G is called a **matching** of G. A matching of G with the maximum number of vertices is called a **maximum matching** of G, while the **matching number** $\nu(G)$ denotes the number of edges in a maximum matching of G. A **perfect matching** of G, if it exists, is a matching of G containing all the vertices of G. The 3–regular graph G_9 in Figure 2.8(a) possesses a perfect matching, shown in Figure 2.8(b).

Let G be a graph of order p with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and let $S = \{u_1, u_2, \ldots, u_p\}$ be a set of vertices disjoint from V. The **corona** of G may be defined as

(a) G_9, *a 3–regular graph.* (b) *A perfect matching for the graph* G_9.

Figure 2.8: *Illustration of (a) a regular graph G_9, and (b) a perfect matching of this graph.*

the graph with vertex set $V \cup S$ and edge set $E(G) \cup \{v_i u_i : \ i = 1, 2, \dots, p\}$. Informally, the corona of G is the graph that is obtained by joining p new vertices to the vertices of G by means of a matching. For the cycle C_4 shown in Figure 2.9(a), the corona of C_4 is shown in Figure 2.9(b).

(a) *The cycle* C_4. (b) *The corona of* C_4.

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Figure 2.9: *Illustration of the corona of a graph.*

A **complete graph** of order p, denoted by K_p, is a graph in which every distinct pair of vertices are adjacent. The complete graph K_p is therefore $(p-1)$–regular. As an illustration of the concept, the complete graphs K_5 and K_6 are shown in Figure 2.10.

A graph G is called n–**partite**, $n \geq 2$, if the vertex set may be partitioned into n subsets, such that no edge of G joins vertices from the same subset. For $n = 2$, G is called **bipartite**, otherwise it is called **multipartite**. The following theorem, a proof of which may be found in [5], pp. 26-27, relates bipartiteness to the occurrence of cycles in a graph.

Theorem 2.3 *A nontrivial graph G is bipartite if and only if it has no odd cycles.* ■

If a vertex in a partition set V_i of a multipartite graph G is adjacent to every vertex in the other sets $\{V_j : j \neq i\}$ for any vertex in G, then G is called **complete** n–**partite**. Such a graph G with $|V_i| = p_i$, $i = 1, 2, \dots, n$, is denoted by $K_{p_1, p_2, \dots, p_n}$. If $p_1 = p_2 = \cdots = p_n = p$, say, then G is called a **complete, balanced** n–**partite** graph and denoted by $K_{n \times p}$. Also, the bipartite graph $K_{1,n} \cong K_{n,1}$ is a popular graph, called

(a) *The complete graph K_5.* (b) *The complete graph K_6.*

Figure 2.10: *Illustration of the concept of a complete graph.*

an n–**star**. The one vertex adjacent to all other vertices of the star is called the **centre**. Illustrations of multi– and bipartite graphs are shown in Figure 2.11.

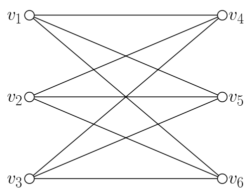
The simplest connected graph structure is known as a **tree**, which is an acyclic connected graph. A graph which is acyclic, is called a **forest**, and consists of a number of disconneced trees. A **leaf** of a tree T is an end–vertex of T. Any vertex adjacent to a leaf is called a **support vertex**, while an r–**support vertex** is a vertex adjacent to at least r leaves. A tree of order 10 is shown in Figure 2.12(a), in which the 5 leaves are indicated as dark vertices. The vertex v_5 is a support vertex and v_8 is a 2–support vertex. A tree is called a **caterpillar** if a path results when all the leaves are removed. If the said path is $P_n : v_1v_2 \cdots v_n$, the caterpillar $C(p_1, p_2, \ldots, p_n)$ is such that v_1 is joined to p_1 leaves, v_2 to p_2 leaves, and so on. An example of the caterpillar $C(3, 1, 2)$ of order 9, with 6 leaves, is shown in Figure 2.12(b). A directed tree is an asymmetric directed graph (a graph with each edge, called an arc, having an associated direction) for which the underlying (undirected) graph is a tree. A directed tree T with a vertex u such that, for every vertex $v \neq u$, there exists a $u - v$ path in T, is called a rooted tree. If T is a rooted tree and w_1w_2 is an arc in T, then w_1 is called the parent of w_2 and w_2 the child of w_1.

Another special type of tree is called a **spider**, which is a number of equally sized paths with one coinciding end–vertex. Denoted by $S_{m\times n}$, the spider consists of m paths of order n, $n \geq 2$, with the centre vertex being the coinciding end–vertex of each path. If the paths are not all of the same length, the graph constructed in this manner is called a **wounded spider** and denoted by $S_{n_1,n_2,\ldots,n_m}$, where $n_i \geq 2$ denotes the order of the i–th path, for $i = 1, 2, \ldots, m$. Examples of the spider graphs $S_{4\times 3}$ and $S_{2,2,3,3}$ are shown in Figure 2.13.

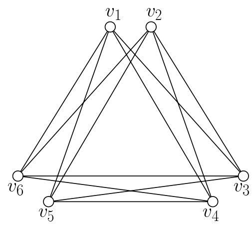
Consider a cycle of length $n \geq 3$, $C_n : v_1v_2 \cdots v_n$, and another vertex, v_0 say. The **wheel** W_n of order n may be defined as the graph join $C_n + \langle v_0 \rangle$, with the vertex v_0 sometimes referred to as the hub. The edges connecting the hub to the rest of the graph are often referred to as spokes. The wheel graphs W_4 and W_5 are shown in Figure 2.14 as examples.

Finally, a **hexagonal graph** $\mathcal{H}_{p,q}$, $p, q \in \mathbb{N}$ (according to [3]) may be defined as the union of the cartesian product graph $P_p \times P_q$, with the edge sets

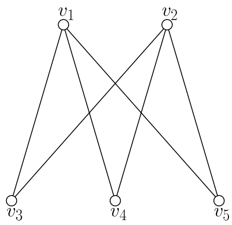
$$\left\{ v_{2i,j}v_{2i-1,j+1} : \; i = 1, 2, \ldots, \left\lceil \frac{p}{2} \right\rceil, \; j = 1, 2, \ldots, q - 1 \right\}$$



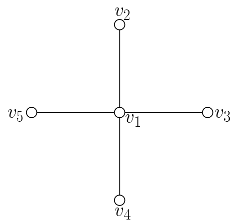
(a) $K_{2\times 3}$.



(b) $K_{3\times 2}$.



(c) $K_{2,3}$.



(d) $K_{1,4}$.

Figure 2.11: *Illustrations of multi– and bipartite graphs.*

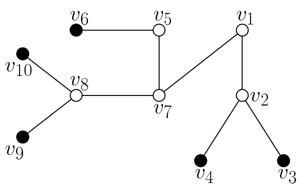
and

$$\left\{v_{2i,j-1}v_{2i+1,j}:\ i=1,2,\ldots,\left\lceil\frac{p}{2}\right\rceil-1,\ j=2,3,\ldots,q\right\}.$$

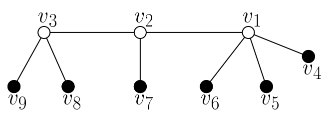
An illustration of such a graph is shown in Figure 2.15, which also indicates the vertex labels.

2.1.6 Independence, Domination and Colourings

A vertex subset $S \subseteq V(G)$ of G is called **independent** if no two vertices in S are adjacent in G. An independent set S of vertices in a graph G is called a **maximal independent** set if S is not a proper subset of any other independent set of G. The maximum cardinal-

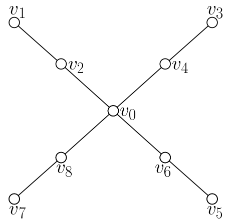


(a) *A tree of order 10 and size 9, with 5 leaves.*

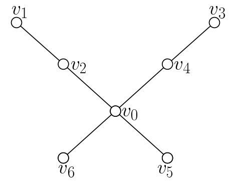


(b) *A caterpillar $C(3,1,2)$ of order 9 and size 8, with 6 leaves.*

Figure 2.12: *Illustrations of trees, with leaves indicated as dark vertices.*

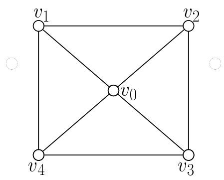


(a) *The spider graph* $S_{4\times 3}$.

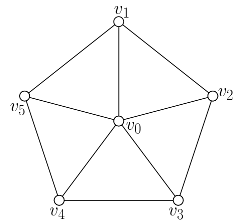


(b) *The wounded spider graph* $S_{2,2,3,3}$.

Figure 2.13*: Illustrations of spiders.*



(a) *The wheel graph* W_4.



(b) *The wheel graph* W_5.

Figure 2.14*: Illustrations of wheels.*

ity of such maximal independent sets S is called the **independence number** of G and is denoted by $\beta(G)$. For the bipartite graph $K_{2,3}$, shown in Figure 2.16(a), both vertex sets $\{v_1, v_2\}$ and $\{v_3, v_4, v_5\}$ are maximal independent sets of $K_{2,3}$. Since the independent set $\{v_3, v_4, v_5\}$, indicated as dark vertices in Figure 2.16(a), is the largest maximal independent set, it follows that $\beta(K_{2,3}) = 3$. Opposite to the notion of independence is the notion of a **clique**, which is a complete subgraph of G that is not an induced subgraph of any other complete subgraph of G, in other words a maximal complete subgraph of G. The maximum order of a clique in G is the so–called **clique number** of G, denoted by $\omega(G)$. The minimum number of cliques into which a graph G may be partitioned is known as the **clique partition number**, $\mathfrak{c}(G)$. For the vertex subset $\{v_1, v_2, v_3, v_6\}$, indicated as dark vertices in the graph G_{10} shown in Figure 2.16(b), the induced graph $\langle v_1, v_2, v_3, v_6\rangle_{G_{10}} \cong K_4$ is the largest clique in the graph G_{10}, so that $\omega(G_{10}) = 4$, while $\mathfrak{c}(G_{10}) = 2$.

A vertex subset $S \subseteq V(G)$ of G is called a **dominating set** if every vertex $v \in V(G)\backslash S$ is adjacent to a vertex $u \in S$. A dominating set S is called a **minimal dominating set** if no proper subset of S is a dominating set. The **lower domination number** (often referred to simply as the domination number), $\gamma(G)$, of a graph G denotes the minimum cardinality of such minimal dominating sets of G. A minimum dominating set of a graph G is therefore often called a $\gamma(G)$–set. The maximum cardinality of a minimal dominating set of G is called the **upper domination number**, $\Gamma(G)$. Proposition 2.1

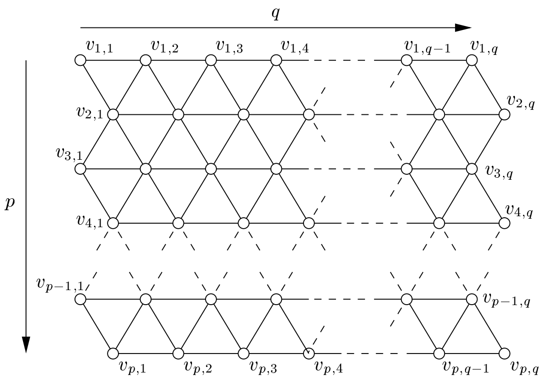
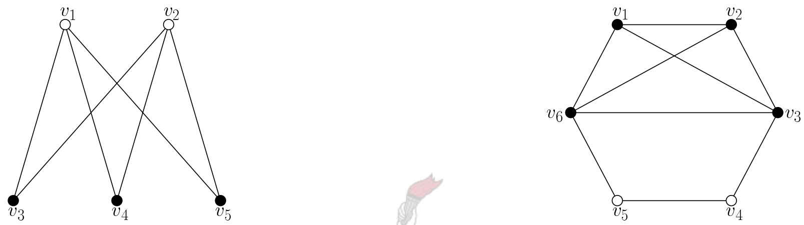


Figure 2.15: *An illustration of the hexagonal graph $\mathcal{H}_{p,q}$.*



(a) *For the graph $K_{2,3}$ $\beta(K_{2,3}) = 3$. A maximum independent set is indicated by the dark vertices.*

(b) *For the graph G_{10}, $\omega(G_{10}) = 4$ and $\mathfrak{c}(G_8) = 2$*

Figure 2.16: *Illustration of (a) independence in a graph $K_{2,3}$, and (b) the notion of a clique in the graph G_{10}.*

states an intuitive result, relating maximal independence and minimal domination of a graph. The reader is referred to [15], pp. 71, for more on this result.

Proposition 2.1 *Every maximal independent set in a graph G is a minimal dominating set of G.* ■

Any dominating set of G that is also independent is called an independent dominating set of G, the minimum cardinality of which is called the **independent domination number**, $i(G)$. For the famous Petersen graph, P in Figure 2.17, the vertex set $\{v_0, v_1, v_2, v_3, v_4\}$ is a minimal dominating set of P of maximal cardinality, yielding $\Gamma(P) = 5$. The domination number of P is, however, $\gamma(P) = 3$, with $\{v_2, v_3, v_5\}$ being a minimum dominating set of P. This set is also a minimum independent dominating set of P, with $i(P) = 3$. In general, Theorem 2.4 gives part of a well–known inequality chain, [15].

Theorem 2.4 *For any graph G, $\gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G)$.* ■

A **colouring** of a graph G is an assignment of colours (or values) to the vertices of G such that no two adjacent vertices have the same colour (value). The minimum number

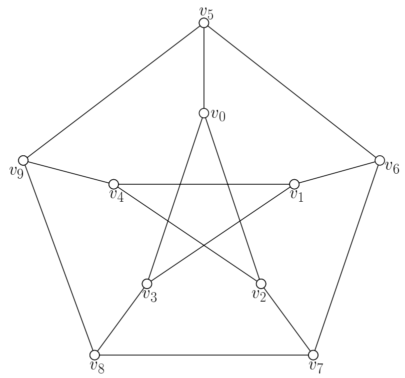


Figure 2.17: *The Petersen graph P, for which: $\gamma(P) = 3$, with $\{v_2, v_3, v_5\}$ being a minimum minimal dominating set of P; $i(P) = 3$, with $\{v_2, v_3, v_5\}$ being a minimum independent dominating set of P; $\Gamma(P) = 5$, with $\{v_0, v_1, v_2, v_3, v_4\}$ being a maximum minimal dominating set of P.*

of colours that may be used for such an assignment is called the **(vertex) chromatic number** of G and is denoted by $\chi(G)$. If $\chi(G) = n$ for a graph G, then the graph is said to be n–chromatic. As an example, the well–known Grötzsch graph, shown in Figure 2.18, is 4–chromatic, with $\{\{v_0, v_2, v_5, v_7\}, \{v_1, v_4, v_6, v_9\}, \{v_3, v_8\}, \{v_{10}\}\}$ representing an optimal set of colour classes.

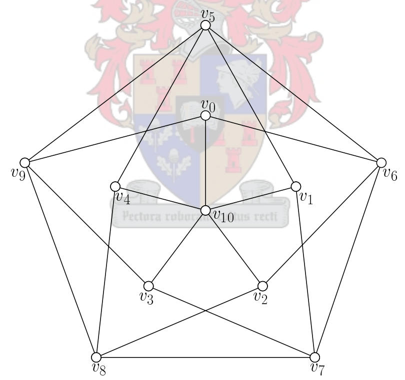


Figure 2.18: *The Grötzsch graph is a 4–chromatic graph, with $\{\{v_0, v_2, v_5, v_7\}, \{v_1, v_4, v_6, v_9\}, \{v_3, v_8\}, \{v_{10}\}\}$ representing an optimal set of colour classes.*

As stated in [13], the intersection of a clique and an independent set of a graph is at most one vertex. Thus, for any graph G it holds that $\omega(G) \leq \chi(G)$ and $\beta(G) \leq \mathfrak{c}(G)$. A graph G is called a **perfect graph** if

(a) $\omega(\langle S \rangle_G) = \chi(\langle S \rangle_G)$ for all $S \subseteq V(G)$, and

(b) $\beta(\langle S \rangle_G) = \mathfrak{c}(\langle S \rangle_G)$ for all $S \subseteq V(G)$.

These conditions are, in fact, equivalent, as discussed in [13]. The well-known **Perfect Graph Theorem** states that a graph is perfect if and only if its complement is perfect. The reader is referred to [29], pp. 291, for a proof of this theorem.

2.2 Basic Concepts in Complexity Theory

Algorithmic complexity is measured by a **time complexity** variable and a **space complexity** variable, usually expressed in terms of the input size n of the algorithm in question. These variables measure respectively the number of basic operations performed, and the memory required by the algorithm. The order of magnitude, denoted by means of the symbol O, of the algorithmic complexity is defined as follows: Let f and g be two real–valued functions. Then $f(n) = O(g(n))$ if there exist a $c \in \mathbb{R}^+$ and an $n_0 \in \mathbb{N}$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$. Informally, the order of magnitude is given by the term growing the fastest as the input size n of the algorithm increases. The function g is said to be an asymptotic upper bound for f. An algorithm for which the order of magnitude of its time complexity is of the form $O(n^k)$, for some $k \in \mathbb{R}^+$ in terms of its input size n, is called a **polynomial time** algorithm. If a problem cannot (with current knowledge) be solved by a polynomial time algorithm, it is referred to as an **intractable** or hard problem, otherwise it is called a **tractable** problem. While the term complexity usually refers to the time complexity of an algorithm, the importance of the space complexity should not be disregarded in practical algorithm implementations.

Decision theory is the branch of complexity theory where the problems to be solved are interpreted as binary questions, that may be answered "yes" or "no". Since any computational problem may be reduced to a decision problem, it is possible, without loss of generality, to consider decision theory only in the theoretical analysis of complexity issues. The class **P** is defined as the set of decision problems that can be solved by way of a polynomial time algorithm. The class **NP** constitutes the set of decision problems of which a solution can be verified in polynomial time, given some additional information. This additional information used to verify the correctness of a solution is called a **certificate**. It is clear that $\mathbf{P} \subseteq \mathbf{NP}$. As an example, consider the following decision problem.

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> **CLIQUE NUMBER**
> **INSTANCE:** A graph G and $k \in \mathbb{N}$.
> **QUESTION:** Does G have clique number $\omega(G) \geq k$?

The following proposition shows that the decision problem CLIQUE NUMBER belongs to the class NP, by using a clique of G of order k, say $\langle v_1, v_2, \ldots, v_k\rangle_G$, as certificate.

Proposition 2.2 *CLIQUE NUMBER* $\in$ *NP*

Proof: The following algorithm verifies whether the induced graph $\langle v_1, v_2, \ldots, v_k\rangle_G$ is a clique in G, a graph of order n, say.

> **Input:** The graph G and vertex set $S = \{v_1, v_2, \ldots, v_k\}$.
> **Step 1:** Test whether $|E(\langle S\rangle_G)| = \frac{1}{2}k(k-1)$. If true, return TRUE. Otherwise, return FALSE.

Note that Step 1 may be completed in $O(k^2n^2)$ time. It is therefore concluded that the algorithm will produce an output in polynomial time. ■

Let L_1 and L_2 be two decision problems. The problem L_1 is *polynomially transformable* to L_2, denoted $L_1 \preceq L_2$, if there exists a mapping f from the instances of L_1 to the instances of L_2, such that

(a) f is computable (deterministically) in polynomial time, and

(b) I is a solution to an instance of L_1 if and only $f(I)$ is a solution to an instance of L_2.

In other words, $L_1 \preceq L_2$ means that an algorithm exists that solves L_1 as a subroutine of an algorithm that solves L_2, with all other operations in the algorithm computable in polynomial time. Informally stated, L_2 is therefore at least as difficult to solve as L_1. The class **NP–complete** is defined as follows.

Definition 2.1 *A decision problem $L \in$ NP–complete if*

(a) $L \in$ NP, and

(b) $L_1 \preceq L$ for all $L_1 \in$ NP. ■

NP–complete problems may be seen as computationally the most difficult problems to solve, since they are at least as difficult to solve as any other problem in NP. Although it is not currently known whether the classes P and NP differ, it has been proven that, if a decision problem L exists for which $L \in$ NP–complete and $L \in$ P, then P = NP, [26]. The well–known satisfiability (SAT) problem serves as an example of an NP–complete problem. In order to describe this problem, the following terminology is introduced.

A **clause** is a boolean expression involving one or more boolean variables (variables with values 0 or 1) conjoined by means of the boolean operation OR. This operation is denoted by $\vee$, as in the example $x_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_4$, where $\overline{x}$ denotes the complement of the boolean variable x. A boolean expression is said to be in **conjunctive normal form**, called a **cnf–formula**, if it comprises several clauses conjoined with the AND operation, denoted by $\wedge$. Definitions of the two boolean operations OR and AND, as well as the complement of a variable, are shown in Tables 2.1 and 2.2.

a	$\overline{a}$
0	1
1	0

Table 2.1: *Definition of the boolean complement.*

An example of a cnf–formula is $x_1 \wedge (x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_3)$. A boolean expression in conjunctive normal form is called a 3cnf–formula if each clause consists of exactly 3 variables, for example

$$(x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_3 \vee x_5 \vee \overline{x}_6).$$

a	b	$a \vee b$	$a \wedge b$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	1	1

Table 2.2: *Definition of the binary operators OR ($\vee$) and AND ($\wedge$).*

A boolean expression is said to be **satisfiable** if an assignment of values for the boolean variables exist for which the expression evaluates to 1. Two satisfiability problems are stated below, and are known to be NP–complete. The reader is referred to [9], or [26] pp. 254–260, for proof of these results.

Satisfiability (SAT)
INSTANCE: A cnf–formula $f(x_1, \ldots, x_n)$, $n \in \mathbb{N}$.
QUESTION: Does an assignment of values to the boolean variables $x_1, \ldots, x_n$ exist for which f will evaluate to 1?

3–Satisfiability (3SAT)
INSTANCE: A 3cnf–formula $f(x_1, \ldots, x_n)$, $n \in \mathbb{N}$.
QUESTION: Does an assignment of values to the boolean variables $x_1, \ldots, x_n$ exist for which f will evaluate to 1?

The following result follows immediately from the definition of NP–completeness stated in Definition 2.1.

Proposition 2.3 *If $L_1 \in$ NP–complete and $L_1 \preceq L_2$, with $L_2 \in$ NP, then $L_2 \in$ NP–complete.* ■

The problem CLIQUE NUMBER will be used as example, to illustrate how a decision problem may be proved to be NP–complete, by mapping an instance of SAT (a known NP–complete problem) in polynomial time to the decision problem in question.

Let ϕ be the cnf–formula

$$\phi = \left(x_1^1 \vee x_2^1 \vee \cdots \vee x_{p_1}^1\right) \wedge \left(x_1^2 \vee x_2^2 \vee \cdots \vee x_{p_2}^2\right) \wedge \cdots \wedge \left(x_1^k \vee x_2^k \vee \cdots \vee x_{p_k}^k\right),$$

consisting of k clauses. The mapping f from ϕ to a graph $f(\phi)$ is defined as follows. The graph $f(\phi)$ is a multipartite graph with k partite sets of cardinalities $p_1, p_2, \ldots, p_k$ respectively. The vertices of the partite set of cardinality p_i are labeled $x_1^i, x_2^i, \ldots, x_{p_i}^i$. The edge set of the graph $f(\phi)$ is the same as that of the corresponding complete multipartite graph, except for the edges between contradictory labels, i.e. labels of which the representative variables in ϕ are complements of each other. For example, if $\phi = (x \vee y) \wedge \overline{x} \wedge (\overline{y} \vee z \vee x)$, the mapping f would result in the graph $f(\phi)$ shown in Figure 2.19.

It is clear that if G is a k–partite graph, then there exists a cnf–formula ϕ with k clauses, such that $f(\phi)$ is isomorphic to the graph G (i.e. identical in structure). The following lemma shows that the mapping f is sufficient to solve the problem CLIQUE NUMBER. The proof is similar to that in [26], pp. 251–253.

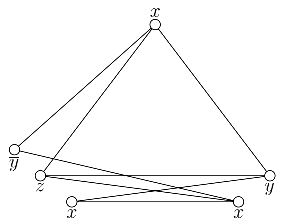

Figure 2.19: *The graph $f(\phi)$ attained from the mapping f, with $\phi = (x \vee y) \wedge \overline{x} \wedge (\overline{y} \vee z \vee x)$.*

Lemma 2.1 *Let G be any graph, suppose $k \in \mathbb{N}$, let f be the mapping defined above, and ϕ be a cnf–formula with k clauses, such that $f(\phi) \cong G$. Then ϕ is satisfiable if and only if $\omega(G) \geq k$.*

Proof: Suppose ϕ has a satisfying assignment of boolean variables. In that satisfying assignment, at least one variable in each clause is assigned the value 1. In each clause of ϕ, select a variable with an assignment of 1 and consider the vertices of G corresponding to these variables under the mapping f. The number of vertices selected is k, since ϕ consists of k clauses. Each vertex–label is in a different clause and no two of these are complements of each other, since all has an assignment of 1. Therefore every pair of these selected vertices are adjacent. This selection forms a clique in G of order k and hence $\omega(G) \geq k$.

Suppose $\omega(G) \geq k$. Then G contains a clique of order k. Consider the variables in ϕ corresponding to the vertices of such a clique. It follows that no two of these variables are in the same clause and no two are complements of each other, since otherwise the corresponding vertices would not be adjacent in G. Therefore each clause contains exactly one of the selected variables. Consider an assignment to the boolean variables in ϕ where every variable corresponding to the clique–vertices are assigned the value 1, and the others 0. Such an assignment is always possible, since none of the clique–variables are contradictory. It follows that ϕ is satisfiable. ■

The following theorem may now be used to show that the decision problem CLIQUE NUMBER is NP–complete.

Theorem 2.5 *SAT $\preceq$ CLIQUE NUMBER*

Proof: An algorithm outline is presented, to solve CLIQUE NUMBER for k–partite graphs, $k \in \mathbb{N}$, which employs SAT as a subroutine.

> **Input:** An integer $k \in \mathbb{N}$, $k \geq 2$, and a k–partite graph G.
>
> **Step 1:** Find a cnf–formula ϕ with k clauses, such that $f(\phi) \cong G$, with f as defined in the above discussion.
>
> **Step 3:** If ϕ is satisfiable, return TRUE. Otherwise, return FALSE.

■

It is known that SAT $\in$ NP–complete [26]. From Theorem 2.5 it now also holds that SAT $\preceq$ CLIQUE NUMBER, with CLIQUE NUMBER $\in$ NP, according to Proposition 2.2. Note that Step 1 in the proof of Theorem 2.5 may be completed in polynomial time, since a cnf–formula ϕ with k clauses may be determined by searching the edges of the graph as each vertex is considered. It follows from Proposition 2.3 that CLIQUE NUMBER $\in$ NP–complete. The reader is referred to [26], pp. 223–270, for a more extensive discussion on complexity theory.

2.3 Chapter Summary

In this chapter, the basic concepts of graph theory and complexity theory, relevant to this thesis, were introduced for the benefit of the reader. The appropriate graph theoretic concepts were discussed in §2.1.1–2.1.6. The last section, §2.2, familiarised the reader with the basic concepts in complexity theory.

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Chapter 3

Roman and Secure Domination

A survey of the known literature on Roman, weak Roman, secure and higher order domination is conducted in this chapter.

3.1 Classical Domination

Using notation similar to that in [2, 3], a **guard function** for a graph $G = (V, E)$ may be defined as a mapping $f : V \to \mathbb{N}_0$ such that $f(v)$ denotes the number of guards stationed at a vertex $v \in V$. A guard function partitions the vertex set V into subsets $V_i = \{v \in V : f(v) = i\}$, with $i \in \mathbb{N}_0$. Since there is a one–to–one correspondence between the function f and the ordered partitions $(V_0, V_1, V_2, \ldots)$, a guard function may unambiguously be written as $f = (V_0, V_1, V_2, \ldots)$. The **weight** of a guard function f is denoted by

$$w(f) = \sum_{v \in V} f(v).$$

A guard function f of a graph G is called a **safe guard function** of G if each unoccupied vertex $v \in V_0$ is adjacent to some occupied vertex $u \in V(G) \backslash V_0$. It follows that $f = (V_0, V_1)$ is a safe guard function of G if and only if the set V_1 is a dominating set of G. For this reason, a safe guard function $f = (V_0, V_1)$ of a graph G is called a **dominating function** (DF) of G, with the minimum weight of a DF denoted by

$$\gamma(G) = \min_{\text{DFs}} |V_1|,$$

which is called the **domination number** of G. The reader is referred to §2.1.6 and [15] for known results on this parameter.

3.2 Roman Domination

In 1997, Revelle [24] considered Emperor Constantine's problem of deploying four field armies (referred to as guards in the general context of this thesis) to secure the Roman

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empire. Publishing the results of this study in 2000, Revelle and Rosing [25] employed an integer programming technique consisting of two parts. The problem of determining the Roman domination number for a graph was solved in the first part (called the Set Covering Deployment Problem), while an optimal placement of a limited number of guards for a graph was found in the second part (called the Maximal Covering Deployment Problem). Dantzig cuts were employed in searching for all the different optimal solutions. In addition to the branch and bound approach being computationally expensive, the calculation process had to be repeated for each graph for which the Roman domination number is required.

For this reason, a graph theoretic approach might prove useful, in which results and bounds pertaining to certain classes of graphs might potentially be uncovered. Prompted by the work of Stewart [27] (in which the work of Revelle and Rosing [25] is summarised), Cockayne *et al.* [6, 11] introduced the notion of a Roman dominating function[1].

Definition 3.1 *A* **Roman dominating function** *(RDF) of a graph G is a safe guard function $f = (V_0, V_1, V_2)$ satisfying the condition that every vertex $v \in V_0$ is adjacent to at least one vertex $u \in V_2$. The minimum weight of an RDF of a graph G is denoted by*

$$\gamma_R(G) = \min_{\text{RDFs}}(|V_1| + 2|V_2|)$$

and is called the **Roman domination number** *of G.* ■

Cockayne *et al.* [6, 11], firstly related the Roman domination number to the classical domination number, as described in the following proposition.

Proposition 3.1 *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.* ■

Various other properties of the minimum weight RDF of a graph were discussed in [6, 11], including the fact that $\gamma(G) = \gamma_R(G)$ if and only if $G \cong \overline{K}_n$. Specific values of Roman domination numbers were found for some special graph classes, including the path P_n and cycle C_n, for any $n \in \mathbb{N}$, as well as the complete bipartite graph $K_{m,n}$, $m, n \in \mathbb{N}$. The cartesian product $P_k \times P_n$ was also examined for various values of $k, n \in \mathbb{N}$, while Dreyer [11] devised a linear time algorithm to compute $\gamma_R(P_k \times P_n)$ for fixed values of k. A classification of when $\gamma_R(G) = \gamma(G) + 1$ and when $\gamma_R(G) = \gamma(G) + 2$ for a graph G was given in [6, 11]. A graph G is called a **Roman graph** if $\gamma_R(G) = 2\gamma(G)$, and a characterisation of Roman graphs was given. Lastly, algorithmic aspects of Roman domination were discussed. A linear time algorithm for computing the Roman domination number of any tree was given, which employs a dynamic programming technique. It was also shown that the following decision problem, corresponding to the Roman domination number, is NP–complete:

> **<u>ROMAN DOMINATING FUNCTION</u>**
> **INSTANCE:** A graph $G = (V, E)$ and a positive integer $n \leq |V(G)|$.
> **QUESTION:** Does G have a Roman dominating function $f = (V_0, V_1, V_2)$ with $w(f) \leq n$?

[1]Results surveyed in this chapter are presented using the notation in [2], for the sake of coherency with the notation used in the upcoming chapters of this thesis.

This was achieved by providing a mapping from the well–known NP–complete problem 3SAT (as defined in §2.2) to the problem ROMAN DOMINATING FUNCTION.

The paper [6], as well as the relevant chapter of [11], were concluded with a listing of open problems of special interest to the authors. One of these suggested problems was the characterisation of Roman trees. Henning [16] was able to resolve this question by introducing a family of rooted trees, consisting of all trees that may be constructed by only three specific operations. It was proved that only trees belonging to this family are Roman trees, thus giving the required characterisation.

Cockayne *et al.* [8] achieved further results regarding the Roman domination number $\gamma_R(G)$ of a graph G, finding a general lower bound involving the order and maximum degree, $\Delta(G)$, of G, as given in the following proposition.

Proposition 3.2 *For any graph G with maximum degree $\Delta(G) \geq 1$,*

$$\gamma_R(G) \geq \frac{2n}{\Delta(G)+1}. \qquad \blacksquare$$

Values of γ_R for the complete graph K_n, $n \in \mathbb{N}$, as well as for the complete multipartite graph $K_{p_1,p_2,\ldots,p_t}$, with $p_i \in \mathbb{N}$, $i = 1, 2, \ldots, t$ and $t \geq 3$, were also established. Additionally, both lower and upper bounds on the Roman domination number were achieved for the cartesian products $C_m \times C_k$ and $P_m \times P_k$, as well as an exact value for $\gamma_R(K_m \times K_k)$, for any $m, k \in \mathbb{N}$.

3.3 Weak Roman Domination

Realising the potential of saving Emperor Constantine substantial costs maintaining his field armies by relaxing the defence requirement, Henning and Hedetniemi [17] proposed the notion of so–called **weak Roman domination**. This concept was suggested as an efficient alternative approach when defending a graph against a single attack, and may formally be defined as follows:

Definition 3.2 *A* **weak Roman dominating function** *(WRDF) is a safe guard function $f = (V_0, V_1, V_2)$ with the property that each vertex $v \in V_0$ is adjacent to some vertex $u \in V_1 \cup V_2$ such that*

$$g(s) = \begin{cases} 1, & \text{if } s = v \\ f(u) - 1, & \text{if } s = u \\ f(s), & \text{if } s \in V \backslash \{u, v\} \end{cases}$$

is also a safe guard function. The minimum weight of a WRDF is denoted by

$$\gamma_r(G) = \min_{\text{WRDFs}} \left(|V_1| + 2|V_2| \right),$$

which is called the **weak Roman domination number** *of G.* $\blacksquare$

Informally, a WRDF may be interpreted as a safe deployment of guards (constituting a safe guard function) on a graph, with maximally two guards per vertex, such that for any unoccupied vertex, there exists a neighbouring guard to move to that vertex, with the resulting deployment again being a safe guard function.

Henning and Hedetniemi [17] observed that every RDF in a graph G is also a WRDF of G. With the introduction of the new parameter $\gamma_r(G)$, the inequality chain in Proposition 3.1 was extended to the following:

Proposition 3.3 *For any graph G, $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$.* ■

As a first step in examining the weak Roman domination number $\gamma_r(G)$ of special graph classes, the value of the parameter for paths and cycles of any order was explored, and found to be $\gamma_r(C_n) = \gamma_r(P_n) = \lceil \frac{3n}{7} \rceil$, $n \in \mathbb{N}$.

Examining special cases of the inequality chain in Proposition 3.3, a characterisation of graphs G for which $\gamma_r(G) = \gamma(G)$, was also given. Furthermore, by defining a family $\mathcal{F}$ of forests possessing certain properties, a characterisation of when $\gamma_r(F) = 2\gamma(F)$, for a forest F, was explored and found to be the case if and only if $F \in \mathcal{F}$.

As a conclusion, the complexity of the decision problem corresponding to the weak Roman domination number, as stated below, was shown to be NP–complete, irrespective of the graph under consideration.

> **WEAK ROMAN DOMINATING FUNCTION**
> **INSTANCE:** A graph $G = (V, E)$ and a positive integer $n \leq 2|V(G)|$.
> **QUESTION:** Does G have a weak Roman dominating function $f = (V_0, V_1, V_2)$ with $w(f) \leq n$?

This was achieved by finding a mapping from the following known NP–complete decision problem for the domination number of a graph to the problem WEAK ROMAN DOMINATING FUNCTION.

> **DOMINATING SET**
> **INSTANCE:** A graph $G = (V, E)$ and a positive integer $k \leq |V(G)|$.
> **QUESTION:** Does G have a dominating set of cardinality k or less?

Cockayne *et al.* [8], were able to build on some of the results obtained in [17] for the weak Roman domination number of special graphs. Values of the weak Roman domination number for the complete graph K_n, $n \in \mathbb{N}$, the complete bipartite graph $K_{p,q}$, $p, q \in \mathbb{N}$, and the complete multipartite graph $K_{p_1,p_2,\ldots,p_t}$, with $p_i \in \mathbb{N}$, $i = 1, 2, \ldots, t$ and $t \geq 3$, were found. Furthermore, upper bounds on γ_r were achieved for the cartesian products $P_m \times P_k$ and $C_m \times C_k$, where $m, k \in \mathbb{N}$. Some of the above mentioned results are stated in the following propositions.

Proposition 3.4 *For the complete bipartite graph $K_{p,q}$, $p \leq q$,*

$$\gamma_r(K_{p,q}) = \begin{cases} 2, & p = 1, 2 \text{ and } q > 1 \\ 3, & p = 3 \\ 4, & p \geq 4. \end{cases}$$ ■

Proposition 3.5 *For the graph $K_{p_1,p_2,\ldots,p_t}$, where $t \geq 3$ and $p_1 \leq p_2 \leq \cdots \leq p_t$,*

$$\gamma_r(K_{p_1,p_2,\ldots,p_t}) = \begin{cases} 2, & p_1 = 1, 2 \\ 3, & p_1 \geq 3. \end{cases}$$

■

3.4 Secure Domination

Observing that a minimum weight WRDF does not necessarily have two guards stationed at any vertex, Cockayne *et al.* [8] introduced the notion of **secure domination**.

Definition 3.3 *A* **secure dominating function** *(SDF) is a safe guard function $f = (V_0, V_1)$ with the property that each vertex $v \in V_0$ is adjacent to some vertex $u \in V_1$ such that*

$$g(s) = \begin{cases} 1, & \text{if } s = v \\ 0, & \text{if } s = u \\ f(s), & \text{if } s \in V \backslash \{u, v\} \end{cases}$$

is also a safe guard function. The minimum weight of an SDF is denoted by

$$\gamma_s(G) = \min_{\text{SDFs}} |V_1|,$$

which is called the **secure domination number** *of G.* ■

Informally, an SDF may be interpreted as a safe deployment of guards on a graph, with maximally one guard per vertex, such that for each unoccupied vertex, there exists a neighbouring guard to move to that vertex, with the resulting deployment again being a safe guard function.

Similar to the situation in classical domination, a minimal secure dominating set of a graph G may be defined as a set V_1 of an SDF $f = (V_0, V_1)$ (called a secure dominating set), such that no proper subset of V_1 is a secure dominating set of G. Cockayne *et al.* [8] provided the following characterisation of minimal secure dominating sets.

Theorem 3.1 *Suppose S is a secure dominating set of G and let X and Y be the sets of redundant and irredundant vertices of S respectively. Also, for each $v \in V(G) \backslash S$, let $A(v, S) = \{s \in S \cap N(v) \mid v$ is defended by s relative to $S\}$. Then S is a minimal secure dominating set for G if and only if, for each $x \in X$ with $N(x) \cap X \neq \emptyset$, there exists a $v_x \in V \backslash S$ such that, for each $s \in A(v_x, S) \backslash \{x\}$, either*

(i) there exists a $w \in V \backslash S$ for which $N(w) \cap S = \{s, x\}$ and $v_x \notin N(w)$, or

(ii) $N(x) \cap S = \{s\}$ and v_x is adjacent to s, but not to x. ■

Note that, for a secure dominating set V_1 of a graph, a vertex v is said to be defended by a vertex u if $u \in N(v) \cap V_1$ and $(V_1 \backslash \{u\}) \cup \{v\}$ is dominating. Regarding the secure domination number, $\gamma_s(G)$, of a graph G, the inequality chain in Proposition 3.3 was extended by Cockayne *et al.* [8] to the following:

Proposition 3.6 *For any graph G,*

$$\gamma(G) \leq \gamma_r(G) \leq \begin{cases} \gamma_R(G) \leq 2\gamma(G), \\ \gamma_s(G). \end{cases} \qquad \blacksquare$$

Lower bounds for the secure domination number $\gamma_s(G)$ involving the order and maximum degree of a graph were found by Cockayne *et al.* [8], in the case where the graph G is K_3–free or K_4–free. These bounds were generalised by Cockayne, Favaron and Mynhardt [7] to the result shown in the following theorem.

Theorem 3.2 *Let $\Delta \geq 3$ and $3 \leq t \leq \Delta + 1$. If the graph G has order n, maximum degree Δ and is K_t–free, then*

$$\gamma_s(G) \geq n\frac{2\Delta - 2t + 5}{(\Delta+1)^2 - (t-1)(t-2)}.$$

For all Δ, t satisfying the hypothesis, the bound is attained for infinitely many n. $\blacksquare$

Considering some specific graph classes, values for γ_s were established for the complete graph K_n, $n \in \mathbb{N}$, the complete bipartite graph $K_{p,q}$, $p, q \in \mathbb{N}$, and the complete multipartite graph $K_{p_1,p_2,\ldots,p_t}$, with $p_i \in \mathbb{N}$, $i = 1, 2, \ldots, t$ and $t \geq 3$, as well as for the path P_n and cycle C_n, $n \in \mathbb{N}$. Both lower and upper bounds were found for the secure domination number γ_s of the cartesian products $P_m \times P_k$ and $C_m \times C_k$, $m, k \in \mathbb{N}$, also relating to the weak Roman domination number of these graphs. Some of the above mentioned results are stated in the following propositions.

Proposition 3.7 *For the complete bipartite graph $K_{p,q}$, $p \leq q$,*

$$\gamma_s(K_{p,q}) = \begin{cases} q, & p = 1 \\ 2, & p = 2 \\ 3, & p = 3 \\ 4, & p \geq 4. \end{cases} \qquad \blacksquare$$

Proposition 3.8 *For the graph $K_{p_1,p_2,\ldots,p_t}$, where $t \geq 3$ and $p_1 \leq p_2 \leq \cdots \leq p_t$,*

$$\gamma_s(K_{p_1,p_2,\ldots,p_t}) = \begin{cases} 2, & p_1 = 1, p_2 \leq 2 \\ 2, & p_1 = 2 \\ 3, & \textit{otherwise}. \end{cases} \qquad \blacksquare$$

By utilising the notion of so–called excellence in graphs, Mynhardt *et al.* [22] were able to characterise trees with equal domination and secure domination numbers. A graph G is said to be γ–excellent if each vertex of G is contained in some minimum dominating set of G. Some useful properties of γ–excellent trees were given, to be used in the characterisation of (γ, γ_s)–trees (trees T for which $\gamma(T) = \gamma_s(T)$). Firstly, a characterisation of γ–excellent trees was obtained, by defining four simple operations for constructing trees.

Letting $\mathcal{E}$ denote the class of all trees obtained from the path P_4 by a finite sequence of these operations, a tree T was shown to be γ–excellent if and only if $T \in \{K_1, K_2\} \cup \mathcal{E}$. By defining two additional operations on a tree, a similar result was obtained for the characterisation of (γ, γ_s)–trees. A brief argument was given showing that a graph is a (γ, γ_s)–graph (a graph G for which $\gamma(G) = \gamma_s(G)$) if and only if it is a (γ, γ_r)–graph (a graph G for which $\gamma(G) = \gamma_r(G)$). Therefore, a characterisation of trees T for which $\gamma(T) = \gamma_r(T)$ was also implicitly given.

Cockayne *et al.* [7] were able to relate the secure domination number $\gamma_s(G)$ of a graph G with its matching number $\nu(G)$, by showing that $\gamma_s(G) \leq n - \nu(G)$ for any graph G of order n. This upper bound was also found to be best possible in general. In an attempt to compare the weak Roman domination number with the secure domination number, it was shown that $\gamma_r(G) = \gamma_s(G)$ for any claw–free graph G. Furthermore, $\gamma_s(G) \leq \frac{2}{3}\beta(G)$ if G is claw–free, and $\gamma_s(G) \leq \beta(G)$ if G is also C_5–free. Additional upper bounds were obtained for a connected, claw–free graph G of order n, which incorporates the minimum vertex degree $\delta(G)$ of G. These bounds were found to be $\gamma_s(G) \leq \frac{3n}{\delta+3}$, and $\gamma_s \leq \frac{2n}{\delta+2}$ if G is also C_5–free.

3.5 Higher Order Domination

Dreyer [11] generalised the notion of Roman domination to allow for multiple movements of guards on a graph. Each vertex has a threshold value, indicating the minimum number of guards needed to defend that particular vertex. He defined a **slide** as a transferrence of guards from an occupied vertex v_j to an adjacent vertex v_i, with the requirement that the vertex v_j still be defended after the move (i.e., more than the threshold number of guards remain at v_j). In this context, a graph is said to be **defended** by a deployment of guards, if for any vertex v, there exists a sequence of slides resulting in the defence of v. According to this definition, each vertex involved in this sequence of slides, must be defended as well. It is noted that, although this may be the first investigation of multiple guard movements, the above definition only allows the defence of a single attack. After the sequence of slides resulting in the successful defence of a vertex, it may be impossible to defend another, different vertex.

Building on the domination concepts introduced in the literature on Roman, weak Roman and secure domination discussed in §3.2–§3.4, Burger *et al.* [2], suggested the notion of so–called **finite order domination** in a graph, which allows for the defense of a graph against multiple attacks. The paper opens with the observation that the previous definitions of a WRDF and SDF are so–called *smart* definitions, in the sense that only the existence of a move strategy is ensured, but that it is up to the strategist to find such a strategy. Hence a so–called *foolproof* definition of a WRDF and SDF was introduced in contrast to the smart definitions, which ensures that *any* move strategy will result in a safe guard function.

Definition 3.4 *A* **foolproof weak Roman dominating function** *(FWRDF) is a safe guard function $f = (V_0, V_1, V_2)$ such that, for each $u \in V_1 \cup V_2$ in the (open) neighbourhood of any $v \in V_0$, the function*

$$g(s) = \begin{cases} 1, & \text{if } s = v \\ f(u) - 1, & \text{if } s = u \\ f(s), & \text{if } s \in V \backslash \{u, v\} \end{cases}$$

is also a safe guard function. The minimum weight of an FWRDF is denoted by

$$\gamma_r^*(G) = \min_{\text{FWRDFs}} (|V_1| + 2|V_2|),$$

which is called the **foolproof weak Roman domination number** *of G.* ■

Definition 3.5 *A* **foolproof secure dominating function** *(FSDF) is a safe guard function $f = (V_0, V_1)$ such that, for each $u \in V_1$ in the (open) neighbourhood of any $v \in V_0$, the function*

$$g(s) = \begin{cases} 1, & \text{if } s = v \\ 0, & \text{if } s = u \\ f(s), & \text{if } s \in V \backslash \{u, v\} \end{cases}$$

is also a safe guard function. The minimum weight of an FSDF is denoted by

$$\gamma_s^*(G) = \min_{\text{FSDFs}} |V_1|,$$

which is called the **foolproof secure domination number** *of G.* ■

The notion of a WRDF in Definition 3.2 will henceforth be called a **smart weak Roman dominating function** (SWRDF), with $\gamma_r(G)$ denoting the **smart weak Roman domination number** of G. The notion of an SDF in Definition 3.3 will henceforth be called a **smart secure dominating function** (SSDF), with $\gamma_s(G)$ denoting the **smart secure domination number** of G. It was briefly noted in [2] that the relationships $\gamma_r(G) \leq \gamma_r^*(G)$ and $\gamma_s(G) \leq \gamma_s^*(G)$ trivially hold for any graph G. Moreover, it was shown that $\gamma_r^*(G) \leq \gamma_R(G)$ for any graph G. The inequality chain in Proposition 3.6 was extended to the following:

Proposition 3.9 *For any graph G,*

$$\gamma(G) \leq \gamma_r(G) \leq \begin{cases} \gamma_r^*(G) \leq \gamma_R(G) \leq 2\gamma(G), \\ \gamma_s(G) \leq \gamma_s^*(G). \end{cases}$$ ■

The notion of smart [foolproof] weak Roman and secure domination was generalised, so that safe guard configurations are guaranteed after each of $k \geq 1$ moves. Such a generalisation was achieved through the following four definitions. To cater for the protection of a graph against a sequence of consecutive attacks, a superscript was introduced in the notation of a guard function, indicating the number of attacks already defended against. For some integer $i \in \mathbb{N}$, let $f^{(i)} = (V_0^{(i)}, V_1^{(i)}, \ldots)$ be a guard function of G and $v_i \in V(G)$.

Denote by $f^{(i+1)}$ another guard function formed by $f^{(i)}$ and v_i. If $v_i \in V_0^{(i)}$, then $f^{(i+1)}$ is the guard function obtained from $f^{(i)}$ by the movement of a guard from its position at $u_i \in V(G)\backslash V_0^{(i)}$ along an edge to v_i in response to an attack. If $v_i \in V(G)\backslash V_0^{(i)}$, then no movement is necessary since u_i may be taken as the vertex v_i, and $f^{(i+1)} = f^{(i)}$. The formal definition of $f^{(i+1)}$ is as follows. For $s \in V(G)$,

$$f^{(i+1)}(s) = \begin{cases} f^{(i)}(s) - 1, & \text{if } s = u_i \text{ and } v_i \in V_0^{(i)} \\ 1, & \text{if } s = v_i \text{ and } v_i \in V_0^{(i)} \\ f^{(i)}(s), & \text{if } s \in V\backslash\{u_i, v_i\} \text{ or } v_i \notin V_0^{(i)}. \end{cases}$$

In order to emphasize the movement involved when $v_i \in V_0^{(i)}$, an alternative notation was also adopted, specifically $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$. Note that in the case where v_i is already occupied, the formal definition of $f^{(i+1)}$ gives $f^{(i+1)} = f^{(i)}$, so $u_i = v_i$. Hence the alternative notation can also be used in this case. The above notation will be used in Definitions 3.6–3.9 below. Due to the difficulty of these definitions, some informal and hopefully intuitive discussion will precede them.

Successful defense against a sequence of attacks at vertices $v_0, v_1, \ldots, v_{k-1}$, starting with the guard function $f^{(0)}$, is a recursive process. Firstly, if necessary (i.e. $v_0 \in V_0^{(0)}$), a guard is moved along an edge from $u_0 \in V(G)\backslash V_0^{(0)}$ to v_0, so that $f^{(1)} = \text{move}(f^{(0)}, u_0 \to v_0)$ is a safe guard function of G. Thereafter for each $i = 1, 2, \ldots, k-1$, sequentially, if $v_i \in V_0^{(i)}$, a guard is moved along an edge from $u_i \in V(G)\backslash V_0^{(i)}$ so that $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ is a safe guard function of G. In the case of smart protection, only the existence of a sequence $u_0, u_1, \ldots, u_{k-1}$ satisfying the above conditions is required for successful defense, while in the foolproof case, any sequence $u_i \in N[v_i] \cap (V(G)\backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$, must result in the successful protection of G. The definitions may now be stated formally.

Pectora roborant cultus recti

Definition 3.6 *A* **smart k–weak Roman dominating function** *(k–SWRDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, V_2^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \ldots v_{k-1}$, there exists a sequence of vertices $u_i \in V_1^{(i)} \cup V_2^{(i)}$ in the neighbourhood of v_i such that the functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for all $i = 0, \ldots, k-1$. The minimum weight of a k–SWRDF is denoted by*

$$\gamma_{r,k}(G) = \min_{k-\text{SWRDFs}} \left(|V_1^{(0)}| + 2|V_2^{(0)}|\right),$$

which is called the **smart k–weak Roman domination number** *of G.* ■

Definition 3.7 *A* **foolproof k–weak Roman dominating function** *(k–FWRDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, V_2^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, the functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for any sequence of vertices $u_i \in V_1^{(i)} \cup V_2^{(i)}$ in the neighbourhood of v_i and all $i = 0, \ldots, k-1$. The minimum weight of a k–FWRDF is denoted by*

$$\gamma^*_{r,k}(G) = \min_{k-\text{FWRDFs}} \left(|V_1^{(0)}| + 2|V_2^{(0)}|\right),$$

which is called the **foolproof k–weak Roman domination number** *of G.* ■

Definition 3.8 *A* **smart k–secure dominating function** *(k–SSDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of a graph, with the property that, for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, there exists a sequence of vertices $u_i \in V_1^{(i)}$ in the neighbourhood of v_i such that the functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for all $i = 0, \ldots, k-1$. The minimum weight of a k–SSDF is denoted by*

$$\gamma_{s,k}(G) = \min_{k-\text{SSDFs}} |V_1^{(0)}|,$$

which is called the **smart k–secure domination number** *of G.* ■

Definition 3.9 *A* **foolproof k–secure dominating function** *(k–FSDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, the functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for any sequence of vertices $u_i \in V_1^{(i)}$ in the neighbourhood of v_i and all $i = 0, \ldots, k-1$. The minimum weight of a k–SSDF is denoted by*

$$\gamma^*_{s,k}(G) = \min_{k-\text{FSDFs}} |V_1^{(0)}|,$$

which is called the **foolproof k–secure domination number** *of G.* ■

From these definitions the notion of *higher order* domination is clear, since k vertices, called *problem vertices*, are secured by way of safe guard functions after each move, for some $k \in \mathbb{N}$. The generalisations are such that $\gamma_{r,1}(G) = \gamma_r(G)$, $\gamma^*_{r,1}(G) = \gamma^*_r(G)$, $\gamma_{s,1}(G) = \gamma_s(G)$, $\gamma^*_{s,1}(G) = \gamma^*_s(G)$ for any graph G. Since the case $k = 0$ implies a static configuration, the convention is made that $\gamma_{r,0}(G) = \gamma^*_{r,0}(G) = \gamma_{s,0}(G) = \gamma^*_{s,0}(G) = \gamma(G)$ for any graph G. It was noted that the relationships shown in Proposition 3.10, regarding the foolproof and smart k–weak Roman and k–secure domination parameters, trivially hold for any graph G.

Proposition 3.10 *For any graph G and any $k \in \mathbb{N}_0$, $\gamma_{r,k}(G) \leq \gamma^*_{r,k}(G)$ and $\gamma_{s,k}(G) \leq \gamma^*_{s,k}(G)$.* ■

The following growth relationships, with respect to increasing values of k, were also proved.

Proposition 3.11 *For any graph G and any $k \in \mathbb{N}_0$,*

(a) $\gamma_{r,k}(G) \leq \gamma_{r,k+1}(G)$,

(b) $\gamma^*_{r,k}(G) \leq \gamma^*_{r,k+1}(G)$,

(c) $\gamma_{s,k}(G) \leq \gamma_{s,k+1}(G)$,

(d) $\gamma^*_{s,k}(G) \leq \gamma^*_{s,k+1}(G)$. ■

For the smart finite order domination numbers, results were obtained in [2] stating how graph decomposition (in the sense of considering subgraph structures of a graph) effects these domination numbers.

Proposition 3.12 *For any graph G and any edge $e \in E(G)$, $\gamma_{r,k}(G) \leq \gamma_{r,k}(G-e)$ and $\gamma_{s,k}(G) \leq \gamma_{s,k}(G-e)$, for all $k \in \mathbb{N}_0$.* ■

It was also briefly illustrated why a similar result is not as easy to obtain for the foolproof case.

Yet further generalisations of the current Definitions 3.6–3.9 were suggested by Burger *et al.* [3], for the case when perpetual or eternal security in a graph is required. These generalisations are given below.

Definition 3.10 *A* **smart [foolproof] ∞–weak Roman dominating function** *(∞–SWRDF) [(∞–FWRDF)] is a k–SWRDF [k–FWRDF] in the limit as $k \to \infty$. The minimum weight of an ∞–SWRDF [∞–FWRDF] is denoted by*

$$\gamma_{r,\infty}(G) = \lim_{k\to\infty} \gamma_{r,k}(G) \quad [\gamma^*_{r,\infty}(G) = \lim_{k\to\infty} \gamma^*_{r,k}(G)]$$

and is called the **smart [foolproof] ∞–weak Roman domination number** *of G.* ■

Definition 3.11 *A* **smart [foolproof] ∞–secure dominating function** *(∞–SSDF) [(∞–FSDF)] is a k–SSDF [k–FSDF] in the limit as $k \to \infty$. The minimum weight of an ∞–SSDF [∞–FSDF] is denoted*

$$\gamma_{s,\infty}(G) = \lim_{k\to\infty} \gamma_{s,k}(G) \quad [\gamma^*_{s,\infty}(G) = \lim_{k\to\infty} \gamma^*_{s,k}(G)]$$

and is called the **smart [foolproof] ∞–secure domination number** *of G.* ■

It was shown that these parameters exist for any graph, and also that the smart ∞–weak Roman domination number is, in fact, equal to the smart ∞–secure domination number for any graph G, i.e. $\gamma_{s,\infty}(G) = \gamma_{r,\infty}(G)$. The same is true for the foolproof parameters, with these parameters being explicitly known for any graph G of order n and with minimum degree δ, namely as $\gamma^*_{r,\infty}(G) = \gamma^*_{s,\infty}(G) = n - \delta$. An attempt to find an explicit result for the smart case proved to be more difficult. The r and s subscripts were deemed superfluous in the case of the infinite order parameters, and the smart and foolproof ∞–domination numbers were henceforth denoted by γ_∞ and γ^*_∞ respectively. Finally, the inequality chain in Proposition 3.9 was extended further with the introduction of Definitions 3.10 and 3.11.

Theorem 3.3 *The relationships*

$$\begin{array}{ccccccccc}
\gamma(G) & \leq & \gamma_{r,k}(G) & \leq & \gamma_{s,k}(G) & \leq & \gamma_\infty(G) & \leq & \chi(\overline{G}) \\
 & & \begin{sideset}{}{}\end{sideset}\text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \\
\gamma(G) & \leq & \gamma^*_{r,k}(G) & \leq & \gamma^*_{s,k}(G) & \leq & \gamma^*_\infty(G) & = & n-\delta
\end{array}$$

hold for all $k \in \mathbb{N}$ and any order n graph G with minimum degree δ. ■

Values for the finite order domination numbers defined above were also explored in [2] for various graph classes. For paths, the smart weak Roman, smart secure and foolproof secure finite order domination numbers were determined, as stated in Proposition 3.13.

Proposition 3.13 *For any path P_n,*

(a) $\gamma_{r,k}(P_n) = \gamma_{s,k}(P_n) = \left\lceil \frac{2k+1}{4k+3} n \right\rceil$, for all $k \in \mathbb{N}_0$,

(b) $\gamma^*_{s,k}(P_n) = \begin{cases} \lceil \frac{k+1}{k+3} n \rceil & \text{if } 2 \le k \le n-2 \\ n-1 & \text{if } k \ge n-1. \end{cases}$ ■

It was also conjectured that the corresponding foolproof weak Roman domination numbers are equal to the foolproof secure domination numbers.

Conjecture 3.1 *For any path P_n and any $k \in \mathbb{N}$, $\gamma^*_{r,k}(P_n) = \gamma^*_{s,k}(P_n)$.* ■

Similar to those for paths, results for cycles were also found, as stated in Proposition 3.14.

Proposition 3.14 *For any cycle C_n,*

(a) $\gamma_{r,k}(C_n) = \gamma_{s,k}(C_n) = \left\lceil \frac{2k+1}{4k+3} n \right\rceil$, for all $k \in \mathbb{N}_0$,

(b) $\gamma^*_{s,k}(C_n) = \begin{cases} \lceil \frac{k+1}{k+3} n \rceil, & \text{if } 0 \le k \le n-3 \\ n-2, & \text{if } k \ge n-3. \end{cases}$ ■

Finally, values for both the smart and foolproof secure finite order domination numbers were found for complete bipartite graphs, as shown in Proposition 3.15.

Proposition 3.15 *For the complete bipartite graph $K_{p,q}$,*

$$\gamma_{s,k}(K_{p,q}) = \gamma^*_{s,k}(K_{p,q}) = \begin{cases} 4, & k = 1 \text{ and } p \ge 4 \\ 2(k+1), & 1 < k \le \lfloor \frac{p-2}{2} \rfloor \\ p, & \lfloor \frac{p-2}{2} \rfloor + 1 \le k < p \\ q, & k \ge p \end{cases}$$

where $p, q \in \mathbb{N}$, with $p \le q$. ■

In the penultimate section of [3], the infinite order domination parameters were examined for various special graph classes. Once again these values were found for paths and cycles, as summerised in Proposition 3.16.

Proposition 3.16 *For any path P_n,*

(a) $\gamma_\infty(P_n) = \left\lceil \frac{n}{2} \right\rceil$,

(b) $\gamma^*_\infty(P_n) = n - 1$.

For any cycle C_n,

(c) $\gamma_\infty(C_n) = \left\lceil \frac{n}{2} \right\rceil$,

(d) $\gamma^*_\infty(C_n) = n - 2$. ■

Although the values for the finite order parameters could not be found for complete multipartite graphs, this was, in fact, not the case for the infinite higher order parameters.

Proposition 3.17 *For the complete multipartite graph $K_{p_1,p_2,\ldots,p_t}$, with $p_1 \leq p_2 \leq \ldots \leq p_t$,*

$$\gamma_\infty(K_{p_1,p_2,\ldots,p_t}) = \gamma^*_\infty(K_{p_1,p_2,\ldots,p_t}) = p_t,$$

for all $t \geq 2$. ■

Cartesian products of paths, cycles and complete graphs respectively were also considered and the results from [3] are summarised in Proposition 3.18.

Proposition 3.18 *For the complete graphs K_p and K_q, with $p \leq q$,*

(a) $\gamma_\infty(K_p \times K_q) = p$,

(b) $\gamma^*_\infty(K_p \times K_q) = pq - (p + q) + 2$.

For any paths P_p and P_q,

(c) $\gamma_\infty(P_p \times P_q) = \left\lceil \frac{pq}{2} \right\rceil$.

(d) $\gamma^*_\infty(P_p \times P_q) = pq - 2$.

For any cycles C_p and C_q, with $p, q \geq 4$,

(e) $\frac{7pq}{23} \leq \gamma_\infty(C_p \times C_q) \leq \left\lceil \frac{pq}{2} \right\rceil$,

(f) $\gamma^*_\infty(C_p \times C_q) = pq - 4$. ■

Burger *et al.* [3] noted that the lower bound in Proposition 3.18(e) is sharp if $p = 3$ and q is small enough (for example, if $4 \leq q \leq 11$), and conjectured that the upper bound is sharp if both $p, q \geq 4$.

Lastly, the resemblance of higher order domination to a game of strategy was noted in [3]. Since war games are typically played on boards consisting of hexagonal cells, the so–called hexagonal graph $\mathcal{H}_{p,q}$ was introduced, as defined in §2.1, and the infinite higher order domination parameters were explored for this graph.

Proposition 3.19 *For any $p, q \in \mathbb{N}$,*

(a) $\gamma_\infty(\mathcal{H}_{p,q}) = \left\lceil \frac{2q}{3} \right\rceil \frac{p}{2}$ *if p is even.*

(b) $\frac{9pq}{43} \leq \gamma_\infty(\mathcal{H}_{p,q}) \leq \left\lceil \frac{2q}{3} \right\rceil \frac{p-3}{2} + q + 1$ *if p is odd.*

(c) $\gamma^*_\infty(\mathcal{H}_{p,q}) = pq - 2.$ ■

The papers by Burger *et al.* [2, 3] were concluded by noting possible generalisations that may be studied in future. These generalisations will be touched upon in §7.2.

Benecke *et al.* [1] extended the results of Theorem 3.15 to apply to a general complete multipartite graph. These results will be discussed in greater detail in a later section on multipartite graphs.

Henning [18] also considered the smart protection of a graph from k consecutive attacks, independently from Burger *et al.* [2, 3], but with the addition of allowing more than two guards per vertex, calling it k–**Roman domination**.

Definition 3.12 *A k–**Roman dominating function**[2] (kRDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{k+1}^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \ldots v_{k-1}$, there exists a sequence of vertices $u_i \in V(G)\backslash V_0^{(i)}$ in the neighbourhood of v_i such that the functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for all $i = 0, \ldots, k-1$. The minimum weight of a kRDF is denoted by*

$$\gamma_R^k(G) = \min_{\text{kRDFs}} \sum_{i=1}^{k+1} i|V_i^{(0)}|,$$

*which is called the k–**Roman domination number** of G.* ■

Note that, with this definition, the maximum number of guards allowed per vertex depends on the number of moves (or attacks) to be secured. This is not the case with the higher order domination parameters defined in [2] (Definitions 3.6–3.9 in this section). For the special cases of $k = 0$ and $k = 1$, the k–Roman domination number reduces to the classical domination number and the smart weak Roman domination number respectively, i.e. $\gamma_R^0(G) = \gamma(G)$ and $\gamma_R^1(G) = \gamma_{r,1}(G)$ for any graph G.

Henning [18] first related the k–Roman domination number, $\gamma_R^k(G)$ for a graph G, to the classical domination number of the graph, $\gamma(G)$. The result is stated in Proposition 3.20.

[2] A different notation is used here to that in [18], for the sake of consistency with the definitions in this chapter.

Proposition 3.20 *For any graph G and for $k \geq 1$,*

$$\gamma(G) \leq \gamma_R^k(G),$$

with equality if and only if there exists a minimum dominating set S such that, for any sequence $v_1, \ldots, v_k$ of vertices of G, there exists a sequence $S_0, S_1, \ldots, S_k$ of $\gamma(G)$–sets such that $S_0 = S$, and for $i = 1, \ldots, k$, either $v_i \in S_{i-1}$, in which case $S_i = S_{i-1}$, or $v_i \notin S_{i-1}$, in which case there exists a vertex $u_i \in S_{i-1}$ adjacent to v_i and $S_i = (S_{i-1} \backslash \{u_i\}) \cup \{v_i\}$. ■

A characterisation of trees T for which $\gamma_R^k(T) = \gamma(T)$ was given, by defining a specific family $\mathcal{T}$ of trees to cater for the case $k = 1$. It was shown that $\gamma_R^k(T) = \gamma(T)$ if and only if $k = 1$ and $T \in \mathcal{T}$, or $k \geq 1$ and T is the corona of a tree. Graphs with large k–Roman domination numbers were also examined and an upperbound involving the classical domination number was found to be that shown in Proposition 3.21.

Proposition 3.21 *For any graph G and for $k \geq 1$, $\gamma_R^k(G) \leq (k+1)\gamma(G)$.* ■

The sharpness of the above mentioned upperbound was examined, with the end result given in Proposition 3.22.

Proposition 3.22 *If F is a forest with a unique $\gamma(F)$–set S, and if F has a component with no $(k+1)$–support vertex, then $\gamma_R^k(F) < (k+1)\gamma(F)$.* ■

Using this result, a family $\mathcal{F}$ of forests was defined, and it was shown that $\gamma_R^k(F) = (k+1)\gamma(F)$ if and only if $F \in \mathcal{F}$.

As a concluding section, the following decision problem regarding the k–Roman dominating function was introduced and, by finding a mapping from the problem **DOMINATING SET** (stated in §3.3), was shown to be NP–complete, irrespective of the graph being considered.

> **k–ROMAN DOMINATING FUNCTION (kRDF)**
> **INSTANCE:** A graph $G = (V, E)$ and a positive integer $n \leq (k+1)|V(G)|$.
> **QUESTION:** Does G have a kRDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{k+1}^{(0)})$ with $w(f^{(0)}) \leq n$?

Burger *et al.* [3] established the following bounds on the infinite order domination number and conjectured that the upper bound is sharp.

Proposition 3.23 *For any graph G, $\beta(G) \leq \gamma_\infty(G) \leq \chi(\overline{G})$.* ■

Burger and Mynhardt [4] found, however, an example in the Grötzsch graph that showed that neither bound in Proposition 3.23 is sharp. Goddard *et al.* [12] were able to provide a similar result which applies to more graphs than just the Grötzsch graph. Calling the parameter γ_∞ the *eternal 1–security number*, they proved the following result.

Proposition 3.24 *For any graph G, if $\beta(G) = 2$, then $\gamma_\infty(G) \leq 3$.* ■

Goddard *et al.* [12] also generalised the current notion of eternal protection of a graph by introducing the so–called *eternal m–security number.* This parameter only differs from the ∞–smart domination number in that it allows for any number of guards to move simultaneously when protecting a problem vertex. It is the view of the author that a study of the simpler notion of eternal protection (in the sense of just one guard–move at a time) be thoroughly conducted first, before further generalisations be introduced. Hence this parameter will not be discussed in this thesis.

Only the papers by Burger *et al.* [2, 3] and Henning [18], as well as the thesis by Dreyer [11] and the paper by Goddard *et al.* [12], as discussed in this section, are known by the author to have considered some form of higher order domination. In [2], maximally two guards per vertex were allowed, while in [18], the maximum number of guards allowed per vertex depended on the number of moves to be secured. It is the aim in the upcoming chapters to provide a more general setting in which the above mentioned concepts, introduced in [2, 3, 18], are special cases. Properties of these generalised higher order domination parameters will be explored, thereby providing deeper insight into the general notion of higher order domination.

3.6 Chapter Summary

A survey of the known literature on topics related to the protection of graphs was given in this chapter. The chapter was divided into four sections, indicating the chronological development of these domination concepts. In §3.2, work done on the notion of Roman domination of graphs in [6, 8, 16], was discussed. This was followed, in §3.3, by a survey of the known literature on weak Roman domination, as conducted in [8, 17]. The notion of secure domination was explored in [8, 22], and the results of these papers are surveyed in §3.4. Finally, the above mentioned domination parameters were generalised by introducing the notion of higher order domination, as established in [2, 3] and [18] independently. The last section, §3.5, surveys work done in these papers in greater detail, for the sake of efficient referencing in chapters to follow. The conclusion of §3.5 mentions that a more general setting for the discussed higher order domination parameters is possible. An establishment and exploration of this setting is conducted in the remainder of this thesis, namely in Chapters 4–6.

Chapter 4

Finite Higher Order Domination

In this chapter, the definitions of finite order domination reviewed in §3.5, are generalised in §4.1 to allow an arbitrary number of guards to be stationed at a vertex. Growth properties of the generalised parameters are examined in §4.2, while the effects of graph decomposition by means of removing an edge is examined in §4.3. The benefits (in the sense of a decreased parameter value) of stationing multiple guards at a vertex, is examined in §4.4, while the complexity of the generalised parameters is considered in §4.5.

4.1 A Framework for Higher Order Domination

As mentioned in §1.2, a significant difference exists between the notion of a dominating function and Roman dominating function on the one hand, and a weak Roman dominating function and secure dominating function on the other. The former two, introduced in §3.1 and Definition 3.1, are static in nature, while the latter two, introduced in Definitions 3.2 and 3.3, posess a dynamic characteristic. It is this characteristic, wherein a *guard* moves from an occupied vertex to an unoccupied vertex, that lead to the notion of finite order domination, as introduced by Burger *et al.* [2] and described in §3.5.

The definitions of finite order domination, Definitions 3.6–3.9, are restricted to maximally 1 or 2 guards per vertex. Although the notion of a k–Roman dominating function, introduced in Definition 3.12, does attempt to relax this restriction, the maximum number of guards allowed on a vertex and the number of attacks to be defended against are not independent of each other. A generalisation to a maximum of ℓ, say, guards per vertex, irrespective of the number of moves k, say, is conducted in this chapter. The following definitions attempt to create such a generalised setting for the exploration of higher order domination.

As mentioned in §3.1, a **guard function** for a graph $G = (V, E)$ may be defined as a mapping $f : V \to \mathbb{N}_0$ such that $f(v)$ denotes the number of guards stationed at a vertex $v \in V$. A guard function partitions the vertex set V into subsets $V_i = \{v \in V : f(v) = i\}$, with $i \in \mathbb{N}_0$. Since there is a one–to–one correspondence between the function f and the ordered partitions $(V_0, V_1, V_2, \ldots)$, a guard function may unambiguously be written as $f = (V_0, V_1, V_2, \ldots)$. The **weight** of a guard function f is denoted by $w(f) = \sum_{v \in V} f(v)$.

For a set $S \subseteq V(G)$ the weight of the guard function f on S is denoted by $f(S) = \sum_{v \in S} f(v)$. A guard function f of a graph G is called a **safe guard function** of G if each unoccupied vertex $v \in V_0$ is adjacent to some occupied vertex $u \in V(G) \backslash V_0$. It follows that $f = (V_0, V_1)$ is a safe guard function of G if and only if the set V_1 is a dominating set of G.

To cater for the protection of a graph against a sequence of consecutive attacks, a superscript is used in the notation of a guard function, indicating the number of attacks already defended against. For some integer i and guard function $f^{(i)} = (V_0^{(i)}, V_1^{(i)}, \ldots)$, the guard function $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ resulting from the movement of a guard stationed at a vertex u_i to an attacked vertex v_i, is determined by

$$f^{(i+1)}(s) = \begin{cases} f^{(i)}(s) - 1, & \text{if } s = u_i \text{ and } v_i \in V_0^{(i)} \\ 1, & \text{if } s = v_i \text{ and } v_i \in V_0^{(i)} \\ f^{(i)}(s), & \text{if } s \in V \backslash \{u_i, v_i\} \text{ or } v_i \notin V_0^{(i)} \end{cases}$$

for $s \in V(G)$.

Definition 4.1 *Let $k, \ell \in \mathbb{N}$. A* **smart k^{th}–order ℓ–dominating function** *((ℓ,k)–SDF) of a graph G is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, there exists a sequence of vertices $u_i \in N[v_i] \cap (V(G) \backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$, such that the guard functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ are safe guard functions for all $i = 0, 1, \ldots, k-1$. The minimum weight of an (ℓ,k)–SDF is denoted by*

$$\gamma_{\ell,k}(G) = \min_{(\ell,k)-\text{SDFs}} \left(\sum_{j=1}^{\ell} j \left| V_j^{(0)} \right| \right)$$

and is called the **smart k^{th}–order ℓ–domination number** *of G.* ■

Each vertex v_i to be defended is called a **problem vertex**, while the sequence of vertices $v_0, v_1, \ldots, v_{k-1}$ is called a **problem sequence**. It is noted that if $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ,k)–SDF, then $|V_j^{(i)}| \geq |V_j^{(i+1)}|$ for all $i = 0, 1, \ldots, k-1$ and $j = 2, 3 \ldots, \ell$, meaning that the number of guards on an already occupied vertex can never increase by a guard movement. In addition to the definition, the case $k = 0$ is allowed as a special convention. In this case there are no problem vertices and hence the configuration $f^{(0)}$ remains static (i.e. there are no moves), which means that $f^{(0)}$ must be a dominating function in the classical sense. Hence $\gamma(G) = \gamma_{1,0}(G)$ for any graph G.

The above definition may informally be interpreted as follows. For a guard function of a graph to be a smart k^{th}–order ℓ–dominating function, it has to, first of all, be a safe guard function with maximally ℓ guards per vertex. Additionally, for any unoccupied vertex, there has to exist a neighbouring occupied vertex such that moving a guard from this vertex to the unoccupied vertex, results in a safe guard function. The same has to hold for one of these new guard configurations. Repeating this requirement for $k \in \mathbb{N}$ such moves means that the original safe guard function is a smart k^{th}–order ℓ–dominating function of the graph.

According to Definition 4.1, if $f^{(0)}$ is an (ℓ, k)–SDF of G, then for any vertex sequence $v_0, v_1, \ldots, v_{k-1}$ of G, there exists a sequence $u_i \in N[v_i] \cap V(G) \backslash V_0^{(i)}$, $i = 0, 1, \ldots, k-1$, such that $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ are all safe guard functions of G. If this is the case, the sequence u_i, $i = 0, 1, \ldots, k-1$, is said to **protect** v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. In some of the proofs in this thesis, this notation serves as a more intuitive presentation of the arguments, since the functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$, $i = 0, 1, \ldots, k-1$, are completely determined by the sequences u_i and v_i, $i = 0, 1, \ldots, k-1$, and the initial deployment $f^{(0)}$.

As an example, Figure 4.1(a) (with dark vertices denoting guard occupation) shows a $(1,1)$–SDF $f^{(0)}$ for the path P_7, which is not a $(1,2)$–SDF. This can be seen by noting that for the problem sequence $\{v_1, v_5\}$ there exist no possible guard moves that will result in a safe guard function $f^{(2)}$. Figure 4.1(b) (with the 2–guard vertex clearly indicated) shows a $(2,2)$–SDF which is actually also a $(2,k)$–SDF for any $k \in \mathbb{N}$. This means that for any problem sequence of any length, there exists guard moves that will result in safe guard functions after each move. Both these configurations are, in fact, minimal, resulting in the parameter values $\gamma_{1,1}(P_7) = 3$ and $\gamma_{2,k}(P_7) = 4$ for all $k \in \mathbb{N}$.

(a) *A minimum* $(1,1)$*–SDF configuration for the graph* P_7*.*

(b) *A minimum* $(2,2)$*–SDF configuration for the graph* P_7*.*

Figure 4.1: *Examples of guard configurations for (a) a minimum* $(1,1)$*–SDF for the graph* P_7*, and (b) a minimum* $(2,2)$*–SDF for the same graph* P_7*.*

The foolproof equivalent to Definition 4.1 is stated as follows:

Definition 4.2 *Let* $k, \ell \in \mathbb{N}$*. A* **foolproof** k^{th}**–order** ℓ**–dominating function** *(*(ℓ, k)*–FDF) of a graph* G *is a safe guard function* $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ *such that, for any sequence of vertices* $v_0, v_1, \ldots, v_{k-1}$*, the guard functions* $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ *are safe guard functions for any sequence of vertices* $u_i \in N[v_i] \cap (V(G) \backslash V_0^{(i)})$ *for all* $i = 0, 1, \ldots, k-1$*. The minimum weight of an* (ℓ, k)*–FDF is denoted by*

$$\gamma^*_{\ell,k}(G) = \min_{(\ell,k)-\text{FDFs}} \left(\sum_{j=1}^{\ell} j \left| V_j^{(0)} \right| \right),$$

which is called the **foolproof** k^{th}**–order** ℓ**–domination number** *of* G*.* ■

As per convention the classical dominating function again results when $k = 0$ and $\ell = 1$, so that $\gamma(G) = \gamma_{1,0}(G) = \gamma^*_{1,0}(G)$. Similarly, the notion of a Roman dominating function, as defined in Definition 3.1, arises as special case of Definitions 4.1 and 4.2 when $k = 0$ and $\ell = 2$, if it is additionally required that any unoccupied vertex $v_i \in V_0^{(0)}$ be adjacent

(a) *A minimum* $(1,1)$*–FDF configuration for the graph* P_7*.*

(b) *A minimum* $(2,1)$*–FDF configuration for the graph* P_7*.*

Figure 4.2: *Examples of guard configurations for (a) a minimum* $(1,1)$*–FDF of the graph* P_7*, and (b) a minimum* $(2,1)$*–FDF of the same graph* P_7*.*

to an occupied vertex $u_i \in V_2^{(0)}$. The notion of a weak Roman dominating function of Definitions 3.2 and 3.4 is included in Definitions 4.1 and 4.2 when $k = 1$ and $\ell = 2$, so that $\gamma_r(G) = \gamma_{1,2}(G)$ and $\gamma_r^*(G) = \gamma_{1,2}^*(G)$, while the notion of a secure dominating function in Definitions 3.3 and 3.5 occurs as special case when $k = 1$ and $\ell = 1$, so that $\gamma_s(G) = \gamma_{1,1}(G)$ and $\gamma_s^*(G) = \gamma_{1,1}^*(G)$.

Definition 4.2 may informally be interpreted as follows. For a guard function to be a foolproof k^{th}–order ℓ–dominating function, it has to be a safe guard function with maximally ℓ guards per vertex. Additionally, for any unoccupied vertex, *any* move to it from a neighbouring occupied vertex has to be a safe guard function as well. The same has to hold for the resulting guard configuration. Repeating this requirement for $k \in \mathbb{N}$ moves means that the original guard function is a foolproof k^{th}–order ℓ–dominating function of the graph.

So, if $f^{(0)}$ is an (ℓ, k)–SDF of G, then for any vertex sequence $v_0, v_1, \ldots, v_{k-1}$ of G, any sequence $u_i \in N[v_i] \cap (V(G) \backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$, **protects** v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$.

The $(1,1)$–SDF shown in Figure 4.1(a) is not a $(1,1)$–FDF for the path P_7, since moving a guard from vertex v_2 to v_3 leaves vertex v_1 undominated. It can easily be verified by way of trial and error that no safe guard function of weight 3 can be a $(1,1)$–FDF of P_7. Figure 4.2(a) shows, however, a $(1,1)$–FDF of weight 4 for the path P_7. Similarly, the $(2,k)$–SDF shown in Figure 4.1(b) is not a $(2,k)$–FDF for the path P_7 for any $k \in \mathbb{N}$, since moving the v_2–guard to v_3 will cause vertex v_1 to be undominated. The safe guard function for P_7 shown in Figure 4.2(b) is, however, a $(2,3)$–FDF, though not a $(2,4)$–FDF. It can also be verified that a $(2,3)$–FDF of P_7 will have a weight of at least 5. Therefore $\gamma_{1,1}^*(P_7) = 4$ and $\gamma_{2,3}^*(P_7) = 5$.

4.2 Growth Properties of Parameters

For the generalised finite order parameters of Definitions 4.1 and 4.2, the inequalities of Proposition 3.10 and Theorem 3.3 may also be generalised accordingly, as shown in Propositions 4.1 and 4.2 respectively.

Proposition 4.1 *For any graph G and any $k, \ell \in \mathbb{N}$, $\gamma_{\ell,k}(G) \leq \gamma^*_{\ell,k}(G)$.*

Proof: If the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ, k)–FDF of G with minimum weight, then, for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, there exists a sequence of vertices $u_i \in N(v_i) \cap (V(G)\backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. Therefore $f^{(0)}$ is also an (ℓ, k)–SDF of G, and the weight of these functions are bounded from below by $\gamma_{\ell,k}(G)$. ■

Proposition 4.2 *For any graph G and any $k, \ell \in \mathbb{N}$,*

(a) $\gamma_{\ell+1,k}(G) \leq \gamma_{\ell,k}(G)$,

(b) $\gamma^*_{\ell+1,k}(G) \leq \gamma^*_{\ell,k}(G)$.

Proof: (a) By Definition 4.1, any (ℓ, k)–SDF of minimum weight is also an $(\ell+1, k)$–SDF, since the set $V_{\ell+1}^{(0)}$ of an $(\ell+1, k)$–SDF may be empty. Since the weights of all $(\ell+1, k)$–SDF's of G are bounded from below by $\gamma_{\ell+1,k}(G)$, the inequality follows.

(b) Similarly, by Definition 4.2, any (ℓ, k)–FDF of minimum weight is also an $(\ell+1, k)$–FDF, since the set $V_{\ell+1}^{(0)}$ of an $(\ell+1, k)$–FDF may be empty. Since the weights of all $(\ell+1, k)$–FDF's of G are bounded from below by $\gamma^*_{\ell+1,k}(G)$, the inequality follows. ■

Simple growth relationships for the parameters in terms of the number of moves, k, trivially follow, by arguments similar to those used in the previous proposition. These arguments are a generalisation of those utilised in [2] for Proposition 3.11.

Proposition 4.3 *For any graph G and any $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$,*

(a) $\gamma_{\ell,k}(G) \leq \gamma_{\ell,k+1}(G)$,

(b) $\gamma^*_{\ell,k}(G) \leq \gamma^*_{\ell,k+1}(G)$.

Proof: (a) From Definition 4.1, it follows that any $(\ell, k+1)$–SDF of G of minimum weight $\gamma_{\ell,k+1}(G)$ is also an (ℓ, k)–SDF of G, and the weight of this last dominating function is bounded from below by $\gamma_{\ell,k}(G)$.

(b) Similarly, from Definition 4.2, it follows that any $(\ell, k+1)$–FDF of G of minimum weight $\gamma^*_{\ell,k+1}(G)$ is also an (ℓ, k)–FDF of G, and the weight of this last dominating function is bounded from below by $\gamma^*_{\ell,k}(G)$. ■

The next result shows that no minimum–weight guard configuration, smart or foolproof, will have more than $k+1$ guards on a vertex, where k is the number of problem vertices that have to be defended. This is to be expected, since a guard vertex with more than $k+1$ guards stationed at it, will not have enough moves (a maximum of k) to move all of the guards away from it. Such a vertex will therefore have unneccessarily many guards, not all required for a minimum deployment configuration. This result may have been the reason why Henning [18] elected to only consider $\ell = k+1$. The same argument is used to prove the proposition for both the smart and foolproof cases.

Proposition 4.4 *For any graph G and any $i, k \in \mathbb{N}$,*

(a) $\gamma_{k+i,k}(G) = \gamma_{k+1,k}(G)$,

(b) $\gamma^*_{k+i,k}(G) = \gamma^*_{k+1,k}(G)$.

Proof: Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{k+i}^{(0)})$ be a minimum weight $(k+i,k)$–SDF [FDF, respectively] of G, $i \geq 2$, but suppose that it is not a $(k+1,k)$–SDF [FDF, respectively] of G. Then $V_{k+j}^{(0)} \neq \emptyset$ for some $2 \leq j \leq i$. Let $v \in V_{k+j}^{(0)}$. Irrespective of the vertex sequence $v_0, v_1, \ldots, v_{k-1}$, $f^{(k)}(v) > 1$, and therefore $f^{(0)}(v)$ is not minimal — a contradiction. It is concluded that $V_{k+j}^{(0)} = \emptyset$ for all $j \geq 2$, and that $f^{(0)}$ is a minimum weight $(k+1,k)$–SDF [FDF, respectively] of G. ■

It follows by the above mentioned result that only a finite number of guards per vertex need to be considered, namely $\ell \in \{1, 2, \ldots, k+1\}$, when exploring results concerning the higher order domination parameters $\gamma_{\ell,k}$ and $\gamma^*_{\ell,k}$.

For the smart parameter, a somewhat less intuitive result involving the maximum degree of a graph, is obtained in Corollary 4.1, which also provides an upper bound on the maximum value of ℓ. The corollary follows from the next theorem, which states that, for a minimum (ℓ,k)–SDF, each occupied vertex will have no more guards than the degree of that vertex, stationed at it. For the proof of this result, the opposite is assumed and a contradiction is obtained by providing a (possibly sub–optimal) protection strategy.

Theorem 4.1 *If $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ,k)–SDF of G with minimum weight $\gamma_{\ell,k}(G)$, then for any $v \in V(G) \backslash V_0^{(0)}$, it holds that $v \in V_m^{(0)}$ with $1 \leq m \leq \deg_G v$.*

Proof: Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ be an (ℓ,k)–SDF of G with minimum weight $w(f^{(0)}) = \gamma_{\ell,k}(G)$. By Definition 4.1 it holds that, for any vertex sequence v_i of G, $i = 0, 1, \ldots, k-1$, there exists a sequence $u_i \in V(G) \backslash V_0^{(i)}$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. Consider any $v \in V(G) \backslash V_0^{(0)}$ and let

$$X^{(i)} = \left(N(v) \cap \left(\bigcup_{j=0}^{i} V_0^{(j)} \right) \right) \cup \{v\},$$

$i = 0, 1, \ldots, k-1$, be the union of $\{v\}$ with the set of vertices $w \in N(v)$ for which $w \in V_0^{(j)}$ for some $0 \leq j \leq i$. Informally stated, the set $X^{(i)}$ consists of the vertex v as well as all the neighbours of v that, at some stage up to this point in the protection strategy, is unoccupied. The composition of $X^{(i)}$ clearly depends on the sequences $v_0, v_1, \ldots, v_i$ and $f^{(0)}, f^{(1)}, \ldots, f^{(i)}$ (i.e. the protection strategy), $i = 0, 1, \ldots, k-1$. Also note that $|X^{(i)}| \leq |N[v]|$ for any vertex sequence $v_0, v_1, \ldots, v_{k-1}$ and any protection strategy.

Suppose $v \in V_m^{(0)}$ with $m > \deg_G v$. The following observations are made:

(a) For any vertex sequence $v_i \in V(G) \backslash N[v]$, $i = 0, 1, \ldots, k-1$, there exists a sequence $u_i \in (V(G) \backslash X^{(0)}) \cap (\cup_{j=1}^{\ell} V_j^{(i)})$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. The reason for this is that no unoccupied neighbour of v is ever a part of the sequence $v_i \in V(G) \backslash N[v]$, $i = 0, 1, \ldots, k-1$.

(b) It holds that $X^{(0)} \cap (\cup_{j=1}^{\ell} V_j^{(0)}) = \{v\}$ and hence $f^{(0)}(X^{(0)} \cap (\cup_{j=1}^{\ell} V_j^{(0)})) = m > \deg_G v$.

It follows that for any sequence $v_i \in V(G)$, $i = 0, 1, \ldots, k-1$, there exists a sequence

$$u_i \in \begin{cases} \left(V(G)\backslash X^{(i)}\right) \cap \left(\bigcup_{j=1}^{\ell} V_j^{(i)}\right) & \text{if } v_i \in V(G)\backslash X^{(i)} \\ X^{(i)} \cap \left(\bigcup_{j=1}^{\ell} V_j^{(i)}\right) & \text{if } v_i \in X^{(i)}, \end{cases}$$

$i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. The validity of this (possibly sub–optimal) protection strategy follows from (a), when $v_i \in V(G)\backslash X^{(i)}$, and from (b) when $v_i \in X^{(i)}$, since the sets $(V(G)\backslash X^{(i)}) \cap (\cup_{j=1}^{\ell} V_j^{(i)})$ and $X^{(i)} \cap (\cup_{j=1}^{\ell} V_j^{(i)})$ are clearly disjoint. However, this protection strategy shows that $f^{(0)}(v)$ is not minimal, since $|X^{(i)} \cap (\cup_{j=1}^{\ell} V_j^{(i)})|$ need not be greater than $\deg_G v < m$ for any $i \in \{0, 1, \ldots, k-1\}$, but is. From this contradiction it follows that $v \in V_m^{(0)}$ with $1 \leq m \leq \deg_G v$. ■

The following result now bounds the number of guards, ℓ, per vertex in any minimum (ℓ, k)–SDF in terms of the maximum degree, Δ, of the graph.

Corollary 4.1 *For any graph G with maximum degree Δ, $\gamma_{\Delta+i,k}(G) = \gamma_{\Delta,k}(G)$, for any $i, k \in \mathbb{N}$.*

Proof: Suppose $i \in \mathbb{N}$ and let $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ be a minimum weight $(\Delta + i, k)$–SDF of G. From Theorem 4.1 it follows that $v \in V_j^{(0)}$, with $1 \leq j \leq \deg_G v \leq \Delta$, for any $v \in V(G)\backslash V_0^{(0)}$, so that $V_j^{(0)} = \emptyset$ for all $j \geq \Delta + 1$. Therefore $f^{(0)}$ is also a minimum weight (Δ, k)–SDF of G. ■

Hence only a finite number of guards $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$ has to be considered when examining the smart parameter $\gamma_{\ell,k}$.

4.3 Effects of Graph Decomposition

When searching for the smart finite order domination parameter, decomposing the graph structure in some sense may prove valuable. The following result, similar to Proposition 3.12, is useful when attempting such simplifications.

Proposition 4.5 *For any graph G and any edge $e \in E(G)$, $\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(G - e)$ for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$.*

Proof: Consider the configuration of an (ℓ, k)–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ of G with minimum weight $w(f^{(0)}) = \gamma_{\ell,k}(G)$. For any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, there exists a sequence of vertices $u_i \in N(v_i) \cap (V(G)\backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. Removing any edge from G may result in

$u_i \notin N(v_i)$ for some i. In this case the value of $w(f^{(0)})$ may be greater than $\gamma_{\ell,k}(G)$, to ensure the existence of a sufficient sequence u_i, $i = 0, 1, \ldots, k-1$. ■

Using this proposition, the following corollary may be proved, as stated in [2].

Corollary 4.2 *If H is a spanning subgraph of a graph G, then $\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(H)$ for all $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$.*

Proof: Let H be a spanning subgraph of G and $J = E(G) \backslash E(H)$ be the edge subset of G not in H. By Proposition 4.5, $\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(G - e)$ for each edge $e \in J$. Since $H \cong G - J$, it follows that $\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(H)$. ■

The following lemma provides another intuitive and useful result.

Lemma 4.1 *Let G be a disconnected graph with components $H_1, H_2, \ldots, H_n$. Then for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$,*

$$\gamma_{\ell,k}(G) = \gamma_{\ell,k}(H_1) + \gamma_{\ell,k}(H_2) + \cdots + \gamma_{\ell,k}(H_n).$$

Proof: No two vertices in different components of G are connected. Let the guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ be an (ℓ, k)–SDF of G with minimum weight $w(f^{(0)}) = \gamma_{\ell,k}(G)$. Consider any component H of G. For any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$ of H, there necessarily exists a sequence $u_i \in N[v_i] \cap (V(G) \backslash V_0^{(i)}) \cap V(H)$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. For the component H, let $V_j^{(0)}(H) = V_j^{(0)} \cap V(H)$ for $j = 0, 1, \ldots, \ell$. Then it follows that the safe guard function $g^{(0)} = (V_0^{(0)}(H), V_1^{(0)}(H), \ldots, V_\ell^{(0)}(H))$ is an (ℓ, k)–SDF for the component H. Therefore

$$\gamma_{\ell,k}(G) \geq \gamma_{\ell,k}(H_1) + \gamma_{\ell,k}(H_2) + \cdots + \gamma_{\ell,k}(H_n). \tag{4.1}$$

Let $g_t^{(0)} = (V_0^{(0)}(H_t), V_1^{(0)}(H_t), \ldots, V_\ell^{(0)}(H_t))$ be a minimum weight (ℓ, k)–SDF of the component H_t, $t = 1, 2, \ldots, n$. Also, let $\tilde{V}_j^{(0)} = V_j^{(0)}(H_1) \cup V_j^{(0)}(H_2) \cup \cdots \cup V_j^{(0)}(H_n)$ for $j = 1, 2, \ldots, \ell$ and consider the safe guard function $\tilde{f}^{(0)} = (\tilde{V}_0^{(0)}, \tilde{V}_1^{(0)}, \ldots, \tilde{V}_\ell^{(0)})$ of G. Then $w(\tilde{f}^{(0)}) = \gamma_{\ell,k}(H_1) + \cdots + \gamma_{\ell,k}(H_n)$, and for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$ of G, there exists a sequence $u_i \in N[v_i] \cap V(G) \backslash \tilde{V}_0^{(i)}$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$, because $u_i \in N(v_i)$ only if $u_i, v_i \in H$ for some component H of G. Therefore it follows that

$$\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(H_1) + \gamma_{\ell,k}(H_2) + \cdots + \gamma_{\ell,k}(H_n). \tag{4.2}$$

By a combination of inequalities (4.1) and (4.2), the desired result follows. ■

An equivalent result holds for the foolproof parameters and can be proved similarly.

Lemma 4.2 *Let G be a disconnected graph with components $H_1, H_2, \ldots, H_n$. Then for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, k+1\}$,*

$$\gamma^*_{\ell,k}(G) = \gamma^*_{\ell,k}(H_1) + \gamma^*_{\ell,k}(H_2) + \cdots + \gamma^*_{\ell,k}(H_n).$$

Proof: No two vertices in different components of G are connected. Let the guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ be an (ℓ, k)–FDF of G with minimum weight $w(f^{(0)}) = \gamma^*_{\ell,k}(G)$. Consider any component H of G. For any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$ of H, any sequence $u_i \in N[v_i] \cap (V(G)\backslash V_0^{(i)}) \cap V(H)$, $i = 0, 1, \ldots, k-1$, protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. For the component H, let $V_j^{(0)}(H) = V_j^{(0)} \cap V(H)$ for $j = 0, 1, \ldots, \ell$. Then it follows that the safe guard function $g^{(0)} = (V_0^{(0)}(H), V_1^{(0)}(H), \ldots, V_\ell^{(0)}(H))$ is an (ℓ, k)–FDF for the component H. Therefore

$$\gamma^*_{\ell,k}(G) \geq \gamma^*_{\ell,k}(H_1) + \gamma^*_{\ell,k}(H_2) + \cdots + \gamma^*_{\ell,k}(H_n). \tag{4.3}$$

Let $g_t^{(0)} = (V_0^{(0)}(H_t), V_1^{(0)}(H_t), \ldots, V_\ell^{(0)}(H_t))$ be a minimum (ℓ, k)–FDF for the component H_t, $t = 1, 2, \ldots, n$. Also, let $\tilde{V}_j^{(0)} = V_j^{(0)}(H_1) \cup V_j^{(0)}(H_2) \cup \cdots \cup V_j^{(0)}(H_n)$ for $j = 1, 2, \ldots, \ell$ and consider the safe guard function $\tilde{f}^{(0)} = (\tilde{V}_0^{(0)}, \tilde{V}_1^{(0)}, \ldots, \tilde{V}_\ell^{(0)})$ of G. Then $w(\tilde{f}^{(0)}) = \gamma^*_{\ell,k}(H_1) + \cdots + \gamma^*_{\ell,k}(H_n)$, and for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, any sequence $u_i \in N[v_i] \cap V(G)\backslash \tilde{V}_0^{(i)}$, $i = 0, 1, \ldots, k-1$, protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$, because $u_i \in N[v_i]$ only if $u_i, v_i \in H$ for some component H of G. Therefore it follows that

$$\gamma^*_{\ell,k}(G) \leq \gamma^*_{\ell,k}(H_1) + \gamma^*_{\ell,k}(H_2) + \cdots + \gamma^*_{\ell,k}(H_n). \tag{4.4}$$

By a combination of inequalities (4.3) and (4.4), the desired result follows. ■

The result of Lemma 4.1 facilitates a concise proof of the next proposition, generalising the corresponding result in [2].

Proposition 4.6 *If the vertex set of a graph G is partitioned into disjoint subsets S_1, S_2,…, S_n, then for all $k \in \mathbb{N}$ and any $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$,*

$$\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(\langle S_1 \rangle) + \gamma_{\ell,k}(\langle S_2 \rangle) + \cdots + \gamma_{\ell,k}(\langle S_n \rangle).$$

Proof: Because the vertex subsets $S_1, S_2, \ldots, S_n$ are disjoint, it follows that $V(\langle S_i \rangle) \cap V(\langle S_j \rangle) = \emptyset$ for all $i \neq j$, $i, j = 1, 2, \ldots, n$. Let $H = \langle S_1 \rangle \cup \langle S_2 \rangle \cup \cdots \cup \langle S_n \rangle$, implying that $\langle S_i \rangle$ is a component of H for every $i = 1, 2, \ldots, n$. Since H is a spanning subgraph of G it follows, by using Corollary 4.2 and Lemma 4.1, that

$$\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(H) = \gamma_{\ell,k}(\langle \mathcal{S}_1 \rangle) + \gamma_{\ell,k}(\langle \mathcal{S}_2 \rangle) + \cdots + \gamma_{\ell,k}(\langle \mathcal{S}_n \rangle). \quad ■$$

Note that in general, it is not possible to establish results similar to those of Proposition 4.5, Corollary 4.2 and Proposition 4.6 for the foolproof parameter $\gamma^*_{\ell,k}(G)$. The result of Proposition 4.5 is crucial in proving Corollary 4.2 and Proposition 4.6, but removing an edge of a graph may increase or decrease the value of the foolproof parameter. An example of the former may be seen by observing that $\gamma^*_{1,1}(K_4) = 1$, while $\gamma^*_{1,1}(K_4 - e) = 2$ for any edge $e \in E(K_4)$, as illustrated in Figures 4.3(a)–(b). Referring to Figures 4.3(c)–(d), it can easily be verified that $\gamma^*_{1,1}(P_4) = 2$, while $\gamma^*_{1,1}(P_1 \cup P_3) = \gamma^*_{1,1}(P_1) + \gamma^*_{1,1}(P_3) = 1+2 = 3$, so that there exists a partition of the vertex set of P_4 into two subsets $\mathcal{S}_1 = V(P_1)$ and $\mathcal{S}_2 = V(P_3)$ for which $\gamma^*_{1,1}(P_4) < \gamma^*_{1,1}(\langle \mathcal{S}_1 \rangle) + \gamma^*_{1,1}(\langle \mathcal{S}_2 \rangle)$. Furthermore, it is a simple task to verify that any partition of $V(P_4)$ satisfies this inequality.

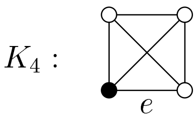
On the other hand, the graph G_1 shown in Figure 4.4(a), serves as a counter–example for the foolproof equivalent of the above mentioned results, since $\gamma^*_{1,1}(G_1) = 6$, yet $\gamma^*_{1,1}(G_1 - e) = 2 + 3 = 5$, as illustrated in Figure 4.4(b). Hence there exists a partition of the vertex set of G_1 into two subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ for which $\gamma^*_{1,1}(G_1) > \gamma^*_{1,1}(\langle \mathcal{S}_1 \rangle) + \gamma^*_{1,1}(\langle \mathcal{S}_2 \rangle)$, contradicting the foolproof equivalent of Proposition 4.6. The dotted edge indicates an alternative graph for which the same holds, showing that the edge e does not necessarily have to be a bridge. Examples that provide a similar counter–example are easy to obtain. It is noted, however, that an example wherein the edge removed is not part of a subgraph isomorphic to $K_{1,3}$, seems less trivial to obtain. Such an example is therefore discussed.

Consider the graph G_2 in Figure 4.5(a) and edge $e \in E(G_2)$ as indicated, so that $G_2 - e \cong H_1 \cup H_2$, as shown in Figure 4.5(b). This figure also shows configurations of the minimum weight $(1,1)$–FDF's $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ and $g^{(0)} = (W_0^{(0)}, W_1^{(0)})$, say, of H_1 and H_2 respectively. The minimum weight $(1,1)$–FDF, $f^{(0)}$, is also unique for H_1, since no other safe guard function of H_1 has weight $\gamma^*_{1,1}(H_1) = 3$. It is also important to note that no $(1,1)$–FDF of H_2 of minimum weight $\gamma^*_{1,1}(H_2) = 5$ will have a guard stationed at vertex v_1. The following paragraph motivates this statement.

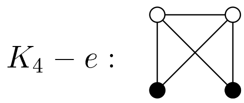
Suppose that this was not the case, and consider a $(1,1)$–FDF $\tilde{g}^{(0)}$ of H_2 of minimum weight $w(\tilde{g}^{(0)}) = 5$, with one of the guards stationed at vertex v_1. The subgraph $\langle v_4, v_5, v_6 \rangle \cong K_3$ may only have one guard and the subgraph $\langle v_8, v_9, \ldots, v_{12} \rangle \cong K_{2,3}$ may only have two guards, since no extra guards are available. The remaining guard has to be placed on either v_3 or v_7, for the resulting guard function $\tilde{g}^{(0)}$ to be a $(1,1)$–FDF of minimum weight. Suppose that the vertex v_3 was occupied by this guard. If the guards commissioned for $\langle v_8, v_9, \ldots, v_{12} \rangle$ occupied two of $\{v_8, v_9, v_{10}\}$, the other vertex of this set would not be dominated. If one of $\{v_8, v_9, v_{10}\}$, v_{10} say, and one of $\{v_{11}, v_{12}\}$, v_{11} say, were occupied, then the guard function $\tilde{g}^{(1)} = \text{move}(\tilde{g}^{(0)}, v_{11} \to v_9)$ would leave v_8 undominated. This means that these two guards have to be stationed on vertices v_{11} and v_{12}. But then the move $\tilde{g}^{(1)} = \text{move}(\tilde{g}^{(0)}, v_3 \to v_2)$ would leave v_7 undominated. So the vertex v_3 is unoccupied and the commissioned guard is stationed at v_7. This, however, means that v_4 cannot be occupied, since $\tilde{g}^{(1)} = \text{move}(\tilde{g}^{(0)}, v_4 \to v_3)$ would leave both v_6 and v_5 undominated. Therefore, the guard commissioned for $\langle v_4, v_5, v_6 \rangle$ occupies either v_5 or v_6. But then the guard function $\tilde{g}^{(1)} = \text{move}(\tilde{g}^{(0)}, v_7 \to v_8)$ will leave v_3 undominated. Since both possibilities (occupation of v_3 or v_7) results in a contradiction, a minimum weight $(1,1)$–FDF of H_2 will necessarily have v_1 unoccupied.

When now examining a possible minimum weight $(1,1)$–FDF of $G_2 \cong H_1 \cup H_2 \cup \langle v_1, v_{13} \rangle$, it can be concluded that $\gamma^*_{1,1}(H_1) + \gamma^*_{1,1}(H_2) = 3 + 5 = 8$ guards are not enough. This can be seen by noting that, because v_1 has to be unoccupied, moving the v_{13}–guard to v_1 leaves v_{14} undominated. Therefore $\gamma^*_{1,1}(G_2) > \gamma^*_{1,1}(H_1) + \gamma^*_{1,1}(H_2) = \gamma^*_{1,1}(G_2 - e)$ by Lemma 4.2, and thus the graph G_2 serves as an appropriate counter–example.

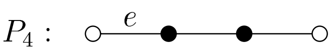
Since the suspicion might still exist that for higher values of ℓ, the maximum number of guards allowed per vertex, or the number of guard–moves k, a foolproof equivalent result to that of Proposition 4.5 may be possible, examples are shown in Figures 4.6 and 4.7 that serves to counter–act this suspicion. Firstly, Figures 4.6(a)–(b) illustrate that $\gamma^*_{2,1}(K_{1,4}) < \gamma^*_{2,1}(K_{1,4} - e)$ for any edge $e \in E(K_{1,4})$, from the symmetry of the graph $K_{1,4}$. The graph G_3 shown in Figures 4.6(c)–(d), however, provides a counter–



(a) *An example of a minimum weight $(1,1)$–FDF of the graph K_4.*



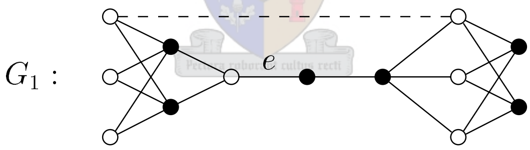
(b) *An example of a minimum weight $(1,1)$–FDF of the graph $K_4 - e$.*



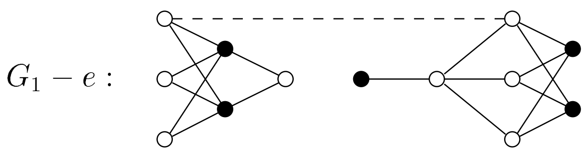
(c) *An example of a minimum weight $(1,1)$–FDF of the graph P_4.*

(d) *An example of a minimum weight $(1,1)$–FDF of the graph $P_4 - e$.*

Figure 4.3: *(a)–(b) The graph K_4 has the property that $\gamma^*_{1,1}(K_4) < \gamma^*_{1,1}(K_4 - e)$ for any edge $e \in E(K_4)$ which is part of a subgraph isomorpic to $K_{1,3}$. (c)–(d) The graph P_4 has the property that $\gamma^*_{1,1}(P_4) \leq \gamma^*_{1,1}(P_4 - e)$ for any edge $e \in E(K_4)$ which is not part of a subgraph isomorpic to $K_{1,3}$.*

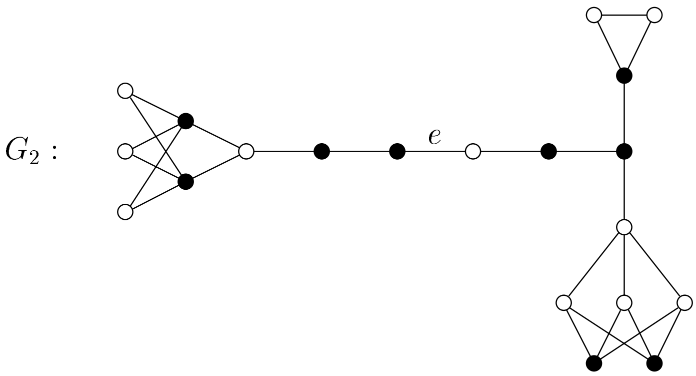


(a) *An example of a minimum weight $(1,1)$–FDF of the graph G_1.*

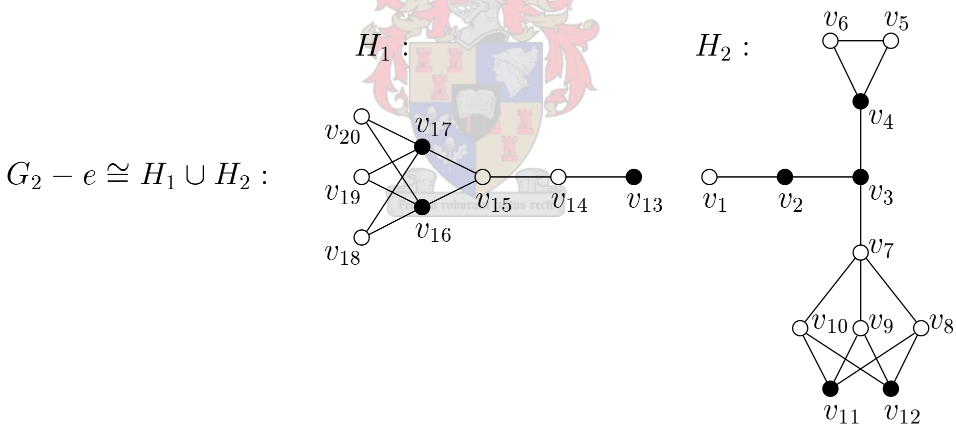


(b) *An example of a minimum weight $(1,1)$–FDF of the graph $G_1 - e$.*

Figure 4.4: *The graph G_1 has the property that there exists an edge $e \in E(G_1)$ which is part of a subgraph isomorpic to $K_{1,3}$, such that $\gamma^*_{1,1}(G_1) > \gamma^*_{1,1}(G_1 - e)$. The dotted edge indicates an alternative graph for which the same holds, showing that the edge e does not necessarily have to be a bridge.*



(a) *An example of a minimum weight* $(1,1)$*–FDF of the graph* G_2*.*

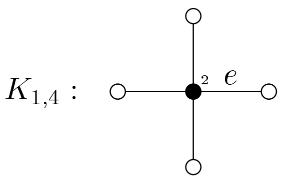


(b) *An example of a minimum weight* $(1,1)$*–FDF of the graph* $G_2 - e$*.*

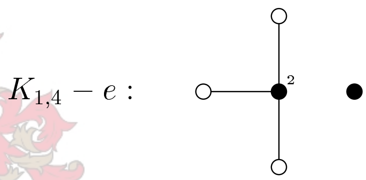
Figure 4.5: *The graph* G_2 *has the property that there exists an edge* $e \in E(G_1)$ *which is not part of a subgraph isomorpic to* $K_{1,3}$*, such that* $\gamma^*_{1,1}(G_2) > \gamma^*_{1,1}(G_2 - e)$*.*

example, since $\gamma^*_{2,1}(G_3) > \gamma^*_{2,1}(G_3 - e)$ for the edge e as indicated. Examining the same concept where more than one, say two, guard–moves are involved, an example is shown in Figures 4.7(a)–(b) that supports the foolproof equivalent to Proposition 4.5, since $\gamma^*_{1,2}(K_4) < \gamma^*_{1,2}(K_4 - e)$ for any edge $e \in E(K_4)$. However, a counter–example, like the graph G_4 shown in Figures 4.7(c)–(d), does exist, since $\gamma^*_{1,2}(G_4) > \gamma^*_{1,2}(G_4 - e)$ for the edge e as indicated.

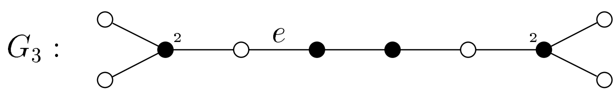
The above mentioned examples suggest that foolproof equivalent results for general graph decomposition might only prove useful when investigating specific graph classes. These examples, however, only serves as a preliminary discussion. For special graph classes, an equivalent result may hold, while the opposite may hold for other classes. A characterisation of such graph classes may provide a significant contribution to understanding the difference between smart and foolproof domination.



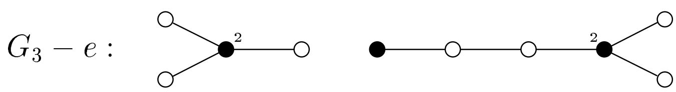
(a) *An example of a minimum weight* $(2,1)$*–FDF of the graph* $K_{1,4}$.



(b) *An example of a minimum weight* $(2,1)$*–FDF of the graph* $K_{1,4} - e$.

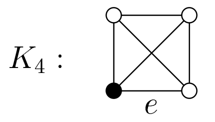


(c) *An example of a minimum weight* $(2,1)$*–FDF of the graph* G_3.

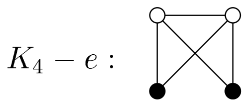


(d) *An example of a minimum weight* $(2,1)$*–FDF of the graph* $G_3 - e$.

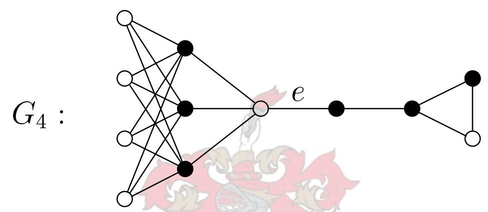
Figure 4.6: *(a)–(b) The graph* $K_{1,4}$ *has the property that* $\gamma^*_{2,1}(K_{1,4}) < \gamma^*_{2,1}(K_{1,4} - e)$ *for any edge* $e \in E(K_{1,4})$. *(c)–(d) The graph* G_3 *has the property that there exists an edge* $e \in E(G_3)$, *such that* $\gamma^*_{2,1}(G_3) > \gamma^*_{2,1}(G_3 - e)$.



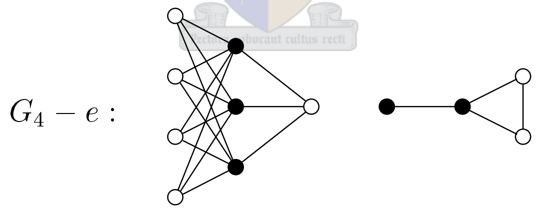
(a) *An example of a minimum weight $(1,2)$–FDF of the graph K_4.*



(b) *An example of a minimum weight $(1,2)$–FDF of the graph $K_4 - e$.*



(c) *An example of a minimum weight $(1,2)$–FDF of the graph G_4.*



(d) *An example of a minimum weight $(1,2)$–FDF of the graph $G_4 - e$.*

Figure 4.7: *(a)–(b) The graph K_4 has the property that $\gamma^*_{1,2}(K_4) < \gamma^*_{1,2}(K_4 - e)$ for any edge $e \in E(K_4)$. (c)–(d) The graph G_4 has the property that there exists an edge $e \in E(G_1)$, such that $\gamma^*_{1,2}(G_4) > \gamma^*_{1,2}(G_4 - e)$.*

4.4 When to Place Multiple Guards at a Vertex

The question of when it is beneficial to place multiple guards at a vertex, seems to be one of the most complicated, yet perhaps one of the most interesting, questions to answer. At the time of writing it was still an unresolved question when (if at all) the parameter value in question will decrease by allowing more guards to be deployed at a vertex. This section serves as an introduction to this investigation, and contains some upper and lower bound results for the smart finite order domination parameters. These results build on the work of Henning [18] and are modified only slightly to cohere with the notation used in this thesis. A lower bound for both the smart and foolproof finite order parameters may easily be obtained, by generalising the corresponding inequalities in Theorem 3.3, obtained by Burger *et al.* [2, 3].

Proposition 4.7 *For any graph G and any $k \in \mathbb{N}$,*

(a) $\gamma(G) \leq \gamma_{\ell,k}(G)$ *for any* $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$,

(b) $\gamma(G) \leq \gamma^*_{\ell,k}(G)$ *for any* $\ell \in \{1, 2, \ldots, k+1\}$.

Proof: Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ be a minimum weight (ℓ, k)–SDF [(ℓ, k)–FDF] of G and consider the set $S = \cup_{j=1}^{\ell} V_j^{(0)}$. Clearly S is a dominating set of G, since $f^{(0)}$ is a safe guard function of G. It follows that $\gamma(G) \leq |S| \leq w(f^{(0)})$. ■

The question of whether the above mentioned bounds are best possible, may be answered immediately for the smart case by generalising the result of Proposition 3.20, proved by Henning [18]. Informally stated, if the smallest number of guards required to protect a graph G against k attacks is $\gamma(G)$, then it is irrelevant how many guards are allowed per vertex. A simplified proof of this result, utilising known results, is presented.

Proposition 4.8 *For any graph G, if $\gamma_{\ell,k}(G) = \gamma(G)$ for some $k \in \mathbb{N}$ and some $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$, then $\gamma_{\ell,k}(G) = \gamma(G)$ for all $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$.*

Proof: Suppose $\gamma_{\ell,k}(G) = \gamma(G)$ for some $\ell \in \{2, 3, \ldots, \min(\Delta, k+1)\}$ and let $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ be a minimum weight (ℓ, k)–SDF of G. Then $S = \cup_{j=1}^{\ell} V_j^{(0)}$ is a dominating set of G and

$$\sum_{j=1}^{\ell} j \left| V_j^{(0)} \right| = \gamma_{\ell,k}(G) = \gamma(G) = |S| = \sum_{j=1}^{\ell} \left| V_j^{(0)} \right|.$$

It follows that $f^{(0)}$ is a $(1, k)$–SDF of G with weight $\gamma(G)$ and hence $\gamma_{1,k}(G) = \gamma(G)$. From Propositions 4.2 and 4.7, $\gamma_{\ell,k}(G) = \gamma(G)$ for any $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$. ■

The corona of a graph is an example of when the bound in Proposition 4.7(a) is sharp. Further properties of graphs for which $\gamma_{\ell,k}(G) = \gamma(G)$ were obtained by Henning [18] and are stated in the next proposition.

Proposition 4.9 *Let G be a graph for which $\gamma_{\ell,k}(G) = \gamma(G)$ for some $k \in \mathbb{N}$ and any $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$. Then for any $(1,k)$–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G with minimum weight $\gamma(G)$,*

(a) $\mathrm{epn}(u, V_1^{(0)}) \cup \{u\}$ *induces a clique for every $u \in V_1^{(0)}$,*

(b) for every $v \in V_0^{(0)}$ that is not a private neighbour of any vertex in $V_1^{(0)}$, there exists a vertex $u \in V_1^{(0)}$ such that $\mathrm{epn}(u, V_1^{(0)}) \cup \{u, v\}$ *induces a clique in G.*

Proof: (a) Let $u \in V_1^{(0)}$ and consider any $v_0 \in \mathrm{epn}(u, V_1^{(0)}) \subseteq V_0^{(0)}$. Then $f^{(1)} = \mathrm{move}(f^{(0)}, u \to v_0)$ is a safe guard function of G. It follows that every vertex in $\mathrm{epn}(u, V_1^{(0)})$ is adjacent to every other vertex in $\mathrm{epn}(u, V_1^{(0)})$.

(b) Consider a vertex $v \in V_0^{(0)}$ that is not a private neighbour of any vertex in $V_1^{(0)}$. Then there exists a $u \in V_1^{(0)}$ such that $f^{(1)} = \mathrm{move}(f^{(0)}, u \to v)$ is safe guard function of G. It follows that v necessarily dominates the set $\mathrm{epn}(u, V_1^{(0)})$. ■

For the case $k = 1$, this condition is actually sufficient, providing a characterisation of graphs G for which $\gamma_{\ell,1}(G) = \gamma(G)$. This result was proved by Henning and Hedetniemi [17] for the case $\ell = 2$, but holds true for any ℓ.

Theorem 4.2 *For any graph G, $\gamma_{\ell,1}(G) = \gamma(G)$ for any $\ell \in \{1, \ldots, \min(\Delta, 2)\}$ if and only if there exists a minimum dominating set S such that*

(a) $\mathrm{epn}(u, S) \cup \{u\}$ *induces a clique for every $u \in S$,*

(b) for every $v \in V(G) \backslash S$ that is not a private neighbour of any vertex in S, there exists a vertex $u \in S$ such that $\mathrm{epn}(u, S) \cup \{u, v\}$ *induces a clique in G.*

Proof: If $\gamma_{\ell,1}(G) = \gamma(G)$, then it follows from Propositions 4.8 and 4.9 that $\gamma_{1,1}(G) = \gamma(G)$ and that there exists a $(1,1)$–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G with weight $w(f^{(0)}) = \gamma(G)$ which satisfies conditions (a) and (b). Therefore there exists a minimum dominating set $S = V_1^{(0)}$ which satisfies these conditions.

Conversely, suppose there exists a minimum dominating set S satisfying conditions (a) and (b), and consider the safe guard function $f^{(0)} = (V(G) \backslash S, S)$ of G. Clearly $f^{(0)}$ is a $(1,1)$–SDF of G with weight $w(f^{(0)}) = \gamma(G)$, so that $\gamma_{1,1}(G) = \gamma(G)$ from Proposition 4.7. It follows from Propositions 4.2 and 4.7 that $\gamma_{\ell,1}(G) = \gamma(G)$ for any $\ell \in \{1, \ldots, \min(\Delta, 2)\}$. ■

From Proposition 4.9, it also follows that if every occupied vertex originally has at most one unoccupied non–private neighbour under the deployment of a minimum weight (ℓ, k)–SDF, and $\gamma_{\ell,k}(G) = \gamma(G)$ for some $k \in \mathbb{N}$, then this is true for any $k \in \mathbb{N}$, as stated in the following corollary.

Corollary 4.3 *Let G be a graph for which $\gamma_{1,k}(G) = \gamma(G)$ for some $k \in \mathbb{N}$, and suppose that there exists a minimum weight $(1,k)$–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G such that $|N(u) \backslash (V_1^{(0)} \cup \mathrm{epn}(u, V_1^{(0)}))| \leq 1$ for every $u \in V_1^{(0)}$. Then $\gamma_{\ell,k}(G) = \gamma(G)$ for any $k \in \mathbb{N}$ and any $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$.*

Proof: The only issue unresolved by Proposition 4.8 is whether the result holds for any $k \in \mathbb{N}$. For any minimum weight $(1, k)$–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G, however, conditions (a) and (b) of Proposition 4.9 hold. From these conditions, as well as the additional requirement on $f^{(0)}$, it clearly follows that any number of $k \in \mathbb{N}$ vertices can be protected under $f^{(0)}$. ■

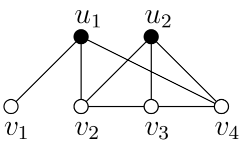


Figure 4.8: *The safe guard function $f^{(0)}$ (indicated by the dark vertices) is an $(\ell, 1)$–SDF of minimum weight of the graph H. The vertex u_2 has two unoccupied non–private neighbours, namely v_2 and v_4. The problem sequence v_1, v_2 cannot be protected under $f^{(0)}$.*

Considering the graph H shown in Figure 4.8, the safe guard function $f^{(0)}$ (indicated by the dark vertices) is an $(\ell, 1)$–SDF of H of minimum weight. The deployment also constitutes a minimum dominating set, and therefore $\gamma_{\ell,1}(H) = \gamma(H)$. The validity of Proposition 4.9 may be verified easily. Since the vertex u_2 has two unoccupied non–private neighbours, namely v_2 and v_4, the conditions for Corollary 4.3 are not met. Note that the problem sequence v_1, v_2 cannot be protected under $f^{(0)}$, and hence $f^{(0)}$ is not an $(\ell, 2)$–SDF of H. However, for the graph $H_1 \cong H \cup \{v_2v_4\}$ (i.e. H with the edge v_2v_4 inserted) $f^{(0)}$ is an (ℓ, k)–SDF of H_1 for any $k \in \mathbb{N}$. This shows that the condition of at most one unoccupied non–private neighbour for every guard–vertex, is sufficient but not necessary.

The above mentioned results now lead to a sufficient condition on graphs G for which $\gamma_{\ell,k}(G) = \gamma(G)$.

Proposition 4.10 *For any graph G, $\gamma_{\ell,k}(G) = \gamma(G)$ for any $k \in \mathbb{N}$ and any $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$ if there exists a minimum dominating set S such that*

(a) $\mathrm{epn}(u, S) \cup \{u\}$ *induces a clique for every $u \in S$,*

(b) for every $v \in V(G) \backslash S$ that is not a private neighbour of any vertex in S, there exists a vertex $u \in S$ such that $\mathrm{epn}(u, S) \cup \{u, v\}$ *induces a clique in G,*

(c) $|N(u) \backslash (S \cup \mathrm{epn}(u, S))| \leq 1$ *for every $u \in S$.*

Proof: Suppose there exists a minimum dominating set S satisfying conditions (a), (b) and (c). Consider the safe guard function $f^{(0)} = (V(G) \backslash S, S)$ of G. Clearly $f^{(0)}$ is a $(1, 1)$–SDF of G, with weight $w(f^{(0)}) = \gamma(G)$, so that $\gamma_{1,1}(G) = \gamma(G)$ from Proposition 4.7. By Corollary 4.3 it follows that $\gamma_{1,k}(G) = \gamma(G)$ for any $k \in \mathbb{N}$. Hence, $\gamma_{\ell,k}(G) = \gamma(G)$ for any $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$, from Propositions 4.2 and 4.7. ■

In the case of the foolproof parameter $\gamma^*_{\ell,k}$, the complete graph K_n, $n \in \mathbb{N}$, is an example of when the bound in Proposition 4.7(b) is sharp, i.e. when $\gamma^*_{\ell,k}(K_n) = \gamma(K_n)$. It is yet unknown whether properties similar to those of the smart parameter, may be obtained in the foolproof case.

Let $\ell_{\max} = \min(\Delta, k+1)$. It follows from Propositions 4.2, 4.4 and Corollary 4.1 that $\gamma_{\ell_{\max},k}(G)$ is the smallest number of guards required to protect the graph G against k consecutive attacks. An upper bound on this value was obtained by Henning [18], as stated in Lemma 3.21, though only for $\ell_{\max} = k+1$, and is stated in the following generalised proposition.

Proposition 4.11 *For any graph G and any $k \in \mathbb{N}$, $\gamma_{\ell_{\max},k}(G) \leq \ell_{\max}\gamma(G)$, with $\ell_{\max} = \min(\Delta, k+1)$.*

Proof: Let $S = \{w_1, w_2, \ldots, w_{\gamma(G)}\}$ be a minimum dominating set of G and consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$ of G, with $V_{\ell_{\max}}^{(0)} = S$, $V_0^{(0)} = V(G)\backslash S$ and $V_j^{(0)} = \emptyset$ for all $j = 1, 2, \ldots, \ell_{\max} - 1$. It is possible to partition $V(G)$ into sets $S_1, \ldots, S_{\gamma(G)}$, such that w_j dominates S_j for $j = 1, 2, \ldots, \gamma(G)$. Since either $f^{(0)}(w_j) = k+1$ or $|S_j| \leq |N[w_j]| \leq \Delta + 1$, $j = 1, 2, \ldots, \gamma(G)$, it follows that for any vertex sequence $v_0, v_1, \ldots, v_{k-1}$ of G, there exists a sequence $u_i \in N[v_i] \cap V(G)\backslash V_0^{(i)}$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. Hence $f^{(0)}$ is an $(\ell_{\max}, k)$–SDF of G. ■

Again the question arises of when equality is obtained in Proposition 4.11. A necessary, but not sufficient, condition for this equality was obtained by Henning [18].

Proposition 4.12 *Let G be a graph for which $\gamma_{\ell_{\max},k}(G) = \ell_{\max}\gamma(G)$ for some $k \in \mathbb{N}$, with $\ell_{\max} = \min(\Delta, k+1)$. Then for every minimum dominating set S and every $v \in S$, the set* $\mathrm{epn}(v, S)$ *contains an independent set of $\ell_{\max}$ vertices.*

Proof: Let $S = \{w_1, w_2, \ldots, w_{\gamma(G)}\}$ be a minimum dominating set of G, but suppose that $\mathrm{epn}(w_1, S)$ contains no set of $\ell_{\max}$ independent vertices. Let $G_1 = \langle \mathrm{epn}(w_1, S) \cup \{w_1\}\rangle$. Then $\beta(G_1) < \ell_{\max}$. Since S is a dominating set of G, the set $V(G)$ may be partitioned into sets $W_1, W_2, \ldots, W_{\gamma(G)}$, such that $W_1 = V(G_1)$ and w_j dominates W_j, $j = 2, 3, \ldots, \gamma(G)$. Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$ of G, with $V_{\ell_{\max}-1}^{(0)} = \{w_1\}$, $V_{\ell_{\max}}^{(0)} = S\backslash\{w_1\}$, $V_0^{(0)} = V(G)\backslash S$ and $V_j^{(0)} = \emptyset$ for all $j = 1, 2, \ldots, \ell_{\max}-2$. For $j \geq 2$, it follows from Proposition 4.4 and Theorem 4.1 that for any vertex sequence $v_0, v_1, \ldots, v_{k-1}$ of $V(G)\backslash W_1$, there exists a sequence $u_i \in N[v_i] \cap (V(G)\backslash V_0^{(i)}) \cap (V(G)\backslash W_1)$, $i = 0, 1 \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$, since either $f^{(0)}(w_j) = k+1$ or $|W_j| \leq |N[w_j]| \leq \Delta+1$. However, since $\beta(G_1) < \ell_{\max}$, it holds that for any vertex sequence $v_0, v_1, \ldots, v_{k-1}$ in W_1, there exists a sequence $u_i \in N[v_i] \cap (V(G)\backslash V_0^{(i)}) \cap W_1$, $i = 0, 1 \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. Therefore $f^{(0)}$ is an $(\ell_{\max}, k)$–SDF of G, so that $\gamma_{\ell_{\max},k}(G) \leq w(f^{(0)}) = \ell_{\max}\gamma(G) - 1$ — a contradiction. ■

Note that if $\gamma_{\ell_{\max},k}(G) = \ell_{\max}\gamma(G)$ for some $k \in \mathbb{N}$, it is unclear when (if at all) this equality is true for all k. Certainly, from Proposition 4.3 it will be true for all $k_1 \geq k$. As mentioned by Henning [18], the necessary condition in Proposition 4.12 is not sufficient, as illustrated in the following observation.

Observation 4.1 *Let $k \in \mathbb{N}$ and consider the $(k+1)$–star graphs $K^{(j)}_{1,k+1}$, each with a leaf denoted by w_j, $j = 1, 2, \ldots, m$ and $m \geq 2$. Let G be the graph resulting from the union of $K^{(j)}_{1,k+1}$ for $j = 1, 2, \ldots, m$, as well as the join of the vertices w_j, such that $\langle w_1, w_2, \ldots, w_m\rangle_G \cong K_m$. Then $\gamma_{\ell_{\max},k}(G) < \ell_{\max}\gamma(G)$, with $\ell_{\max} = \min(\Delta, k+1)$, while for every minimum dominating set S of G and every $v \in S$, the set $\mathrm{epn}(v, S)$ contains an independent set of $\ell_{\max}$ vertices.*

Proof: Let v_j be the centre of the star $K^{(j)}_{1,k+1}$, $j = 1, 2, \ldots, m$. Clearly, the set $S = \{v_1, v_2, \ldots, v_m\}$ is the unique minimum dominating set of G, and for $j = 1, 2, \ldots, m$, the set $\mathrm{epn}(v_j, S) = N(v_j)$ is an independent set of $k+1$ vertices. Consider the safe guard function $f^{(0)} = (V^{(0)}_0, V^{(0)}_1, \ldots, V^{(0)}_{\ell_{\max}})$, with $V^{(0)}_1 = \{w_1\}$, $V^{(0)}_{\ell_{\max}-1} = \{v_j : \ j = 1, 2, \ldots, m\}$ and $V^{(0)}_0 = V(G)\backslash(V^{(0)}_1 \cup V^{(0)}_{\ell_{\max}-1})$. Then $f^{(0)}$ is an $(\ell_{\max}, k)$–SDF of G with weight $w(f^{(0)}) = mk + 1$. Since $\Delta \geq k+1$ for any $k \in \mathbb{N}$, it follows that $\gamma_{\ell_{\max},k}(G) \leq w\left(f^{(0)}\right) = mk + 1 < m(k+1) = \ell_{\max}\gamma(G)$. ■

An example of the graph constructed in Observation 4.1, for the case $m = 4$ and $k = 3$, is shown in Figure 4.9. The unique minimum dominating set is indicated by the dark vertices.

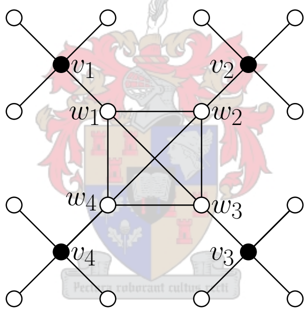


Figure 4.9: *An example of the graph constructed in Observation 4.1, for the case $m = 4$ and $k = 3$. The unique minimum dominating set is indicated by the dark vertices.*

Another open problem is to determine for which graphs G $\gamma_{\ell,k}(G) \leq \ell_{\max}\gamma(G)$ for any graph G, any $k \in \mathbb{N}$ and any $\ell \in \{1, 2, \ldots, \ell_{\max}\}$. If this is true, it would follow that, if $\gamma_{\ell_{\max},k}(G) = \ell_{\max}\gamma(G)$ for some $k \in \mathbb{N}$, then $\gamma_{\ell,k}(G) = \ell_{\max}\gamma(G)$ for any $\ell \in \{1, 2, \ldots, \ell_{\max}\}$, meaning that an increase in the number of guards allowed per vertex does not improve the parameter value. The upper bound may be improved upon in general, if additional requirements are enforced. The following proposition illustrates an example of this, and follows directly from the argument used in Proposition 4.12.

Proposition 4.13 *For any graph G, if there exists a minimum dominating set S of G, such that $N(v)\backslash S$ contains at most ℓ independent vertices for every $v \in S$, $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$, then $\gamma_{\ell,k}(G) \leq \ell\gamma(G)$, for any $k \in \mathbb{N}$.*

Proof: Denote the set S by $\{w_1, w_2, \ldots, w_{\gamma(G)}\}$ and consider the safe guard function $f^{(0)} = (V^{(0)}_0, V^{(0)}_1, \ldots, V^{(0)}_\ell)$ of G, with $V^{(0)}_\ell = S$ and $V^{(0)}_0 = V(G)\backslash S$. The set $V(G)$ may be partitioned into sets $W_1, W_2, \ldots, W_{\gamma(G)}$, such that $w_j \in W_j$ and w_j dominates W_j,

$j = 1, 2, \ldots, \gamma(G)$. Since $\beta(\langle W_j \rangle) \leq \ell$, it follows that for any sequence $v_0, v_1, \ldots, v_{k-1}$ in W_j, there exists a sequence $u_i \in N[v_i] \cap (V(G) \backslash V_0^{(i)}) \cap W_j$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. Therefore $f^{(0)}$ is an (ℓ, k)–SDF of G, and it follows that $\gamma_{\ell,k}(G) \leq w(f^{(0)}) = \ell\gamma(G)$. ■

For graphs G with the property stated in Proposition 4.13, it follows from Proposition 4.2 that $\gamma_{\ell_{\max},k}(G) \leq \gamma_{\ell,k}(G) \leq \ell\gamma(G) \leq \ell_{\max}\gamma(G)$. In this case, $\gamma_{\ell,k}(G)$ has the same value for any ℓ, if the equality $\gamma_{\ell_{\max},k}(G) = \ell_{\max}\gamma(G)$ holds true.

4.5 Complexity

Henning [18] was able to show that the complexity of a decision problem for the determination of a $(k+1, k)$–smart dominating function is NP–complete. The result obtained by him is discussed in this section. As mentioned in §3.3, the following decision problem is known to be NP–complete [17, 18].

DOMINATING SET
INSTANCE: A graph G and a positive integer $s \leq |V(G)|$.
QUESTION: Does G have a dominating set of cardinality s or less?

The following decision problem, depending on the number of guards stationed at a vertex, will henceforth be considered. Note that $\ell_{\max} = \min(\Delta, k+1)$.

(ℓ, k)–SMART DOMINATING FUNCTION
INSTANCE: A graph H and a positive integer $j \leq \ell_{\max}|V(H)|$.
QUESTION: Does H have an (ℓ, k)–SDF of weight j or less?

In order to examine the complexity of this decision problem, it is important to note the following.

Proposition 4.14 *(ℓ, k)–SMART DOMINATING FUNCTION $\in$ NP.*

Proof: The following algorithm outline is presented, to verify whether a specified guard function $f^{(0)}$ of a graph G is an (ℓ, k)–SDF of G with weight j or less.

Input: A graph G, guard function $f^{(0)}$ and an integer $j \leq \ell_{\max}|V(H)|$.
Step 1: Test whether $w(f^{(0)}) \leq j$ and $f^{(0)}(v) \leq \ell$ for every $v \in V(G)$. If true, continue. Otherwise return FALSE.
Step 2: For each problem sequence of length k, verify that $f^{(0)}$ can protect the problem sequence. If so, return TRUE. Otherwise return FALSE.

Since there are $\binom{n}{k} = O(n^k)$ problem sequences to examine, and for each problem sequence at most j^k moves need to be considered, the algorithm will produce an output in polynomial time. ■

The decision problem $(\ell_{\max}, k)$–SMART DOMINATING FUNCTION will now be shown to be NP–complete, by considering the following mapping, as introduced by Henning [18]. For any graph G, let G' be the corona of G. For each end–vertex $v \in V(G')$, add the $(k+1)$–star $K_{1,k+1}$ such that v is joined to exactly one leaf of the star. Let H denote the resulting graph, and let $h : G \rightarrow H$ be the mapping achieving this construction. An example of the graph $h(K_3)$ is shown in Figure 4.10. Also, let $\mathcal{H}$ denote the family of graphs H that can be constructed in this fashion. Note that for any G, the graph H can be constructed in polynomial time, and that $\ell_{\max} = k+1$ when considering H, since $\Delta(H) \geq k+1$.

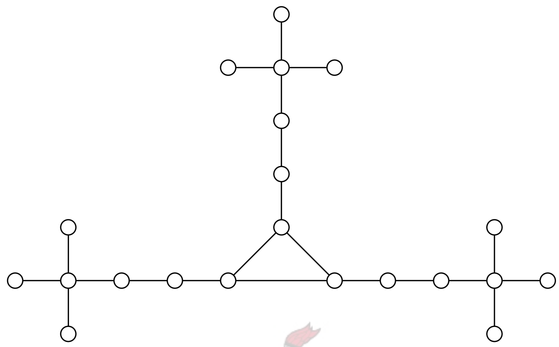


Figure 4.10: *Example of the mapping h for the graph K_3.*

The following lemma and corollary is useful when analysing the complexity of $(\ell_{\max}, k)$–SMART DOMINATING FUNCTION.

Lemma 4.3 *Let G be a graph and $H \cong h(G)$, with h as defined above. Then*

$$\gamma_{k+1,k}(H) = \gamma(G) + (k+1)|V(G)|.$$

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let F_i be the component of $H - V(G)\backslash\{v_i\}$ containing v_i, $i = 1, 2, \ldots, n$. Note that F_i is isomorphic to the graph obtained by joining a leaf of $K_{1,k+1}$ to a vertex of K_2. Let $i \in \{1, 2, \ldots, n\}$ and let v_i, w_i, x_i, y_i denote the path from v_i to the centre y_i of the star. There exists a minimum weight $(k+1, k)$–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{k+1}^{(0)})$ of H such that $f^{(0)}(z) = 0$ for every leaf z adjacent to y_i. It also follows that $f^{(0)}(y_i) \geq k$ and $f^{(0)}(N[w_i]) \geq 1$, so that $f^{(0)}(V(F_i)) \geq k+1$. Let $S = V(G)\backslash V_0^{(0)}$.

Suppose $f^{(0)}(V(F_i)) > k+1$. Then it may be assumed that $f^{(0)}(v_i) \geq 1$, $f^{(0)}(w_i) = 1$, $f^{(0)}(x_i) = 0$, $f^{(0)}(y_i) = k$ and $f^{(0)}(z) = 0$ for every leaf z adjacent to y_i. It follows that $v_i \in S$.

Suppose $f^{(0)}(V(F_i)) = k+1$. Then $f^{(0)}(y_i) = k$ and $f^{(0)}(N[w_i]) = 1$. If $f^{(0)}(x_i) = 1$, then $v_i, w_i \in V_0^{(0)}$, so that v_i is necessarily dominated by S, since $f^{(0)}$ is a safe guard function of H. If $f^{(0)}(x_i) = f^{(0)}(w_i) = 0$, then the protection of the sequence of k distinct vertices in $N(y_i)\backslash\{x_i\}$ necessarily leaves x_i undominated under $f^{(k)}$. Therefore $f^{(0)}(w_i) = 1$ and so $f^{(0)}(v_i) = 0$. Since the protection of the vertex sequence $s_0, s_1, \ldots, s_{k-2}, x_i$, with

$s_j \in N(y_i)\backslash\{x_i\}$, $j = 0, 1, \ldots, k-2$, results in a safe guard function $f^{(c)}$, with $f^{(c)}(w_i) = f^{(c)}(v_i) = 0$, it follows that v_i is also dominated by S under $f^{(0)}$.

Thus, S is a dominating set of G, so that $\gamma(G) \leq |S|$. Furthermore, if $v_i \in S$, then $f^{(0)}(V(F_i)) \geq k+2$, while if $v_i \notin S$, then $f^{(0)}(V(F_i)) \geq k+1$. Therefore,

$$\begin{aligned} \gamma_{k+1,k}(H) &= w(f^{(0)}) \\ &\geq (k+2)|S| + (k+1)(|V(G)| - |S|) \\ &= |S| + (k+1)|V(G)| \\ &\geq \gamma(G) + (k+1)|V(G)|. \end{aligned} \tag{4.5}$$

Let D be a minimum dominating set of G and consider the safe guard function $g^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{k+1}^{(0)})$ of H, with $V_1^{(0)} = D \cup \{w_i : \ i = 1, 2, \ldots, n\}$, $V_k^{(0)} = \{y_i : \ i = 1, 2, \ldots, n\}$ and $V_0^{(0)} = V(H)\backslash(\cup_{j=1}^{k+1} V_j^{(0)})$. Then $g^{(0)}$ is clearly a $(k+1,k)$–SDF of H, and so

$$\begin{aligned} \gamma_{k+1,k}(H) &\leq w(g^{(0)}) \\ &= (k+2)|D| + (k+1)(|V(G)| - |D|) \\ &= |D| + (k+1)|V(G)| \\ &= \gamma(G) + (k+1)|V(G)|. \end{aligned} \tag{4.6}$$

The desired result follows by a combination of (4.5) and (4.5). ■

The following result follows immediately from Lemma 4.3 and shows how the mapping h defined above, is used to prove that the decision problem $(\ell_{\max}, k)$–SMART DOMINATING FUNCTION is NP–complete.

Proposition 4.15 *For any graph G, let $s \in \mathbb{N}$ and $H \cong h(G)$, with h as defined above. With $j = s + (k+1)|V(G)|$, $\gamma(G) \leq s$ if and only if $\gamma_{\ell_{\max},k}(H) \leq j$.*

Proof: The result follows immediately from Lemma 4.3, by using the fact that $\ell_{\max} = k+1$ for any $H \in \mathcal{H}$. ■

Using the mapping h defined above, it is now proved that the decision problem $(\ell_{\max}, k)$–SMART DOMINATING FUNCTION is at least as hard to solve as the problem DOMINATING SET, and hence that it is NP–complete.

Theorem 4.3 $(\ell_{\max}, k)$*–SMART DOMINATING FUNCTION $\in$ NP–complete.*

Proof: An algorithm outline is presented, to solve $(\ell_{\max}, k)$–SMART DOMINATING FUNCTION for graphs belonging to the family $\mathcal{H}$, as defined above, that employs DOMINATING SET as a subroutine.

Input: A graph $H \in \mathcal{H}$ and an integer $j \leq \ell_{\max}|V(H)|$.

Step 1: Find the graph G for which $h(G) \cong H$, with the mapping h as defined previously.
Step 2: Set $s = j - (k+1)|V(H)|$ and solve DOMINATING SET for G and the integer s.
Step 3: Return the solution of Step 2.

From this algorithm it follows that DOMINATING SET $\preceq$ $(\ell_{\max}, k)$–SMART DOMINATING FUNCTION. Therefore, by Proposition 2.3, it holds that $(\ell_{\max}, k)$–SMART DOMINATING FUNCTION $\in$ NP–complete, since DOMINATING SET is a known NP–complete problem. ■

Although Theorem 4.3 only verifies the NP–completeness of the decision problem where no restriction is placed on the number of guards stationed at a vertex, it is expected that the more general decision problem (ℓ, k)–SMART DOMINATING FUNCTION, for some $\ell < \ell_{\max}$, is NP–complete as well, since there is an added restriction on the guard function in this case. A decision problem concerning foolproof finite order domination may also be defined, similarly to the definition of (ℓ, k)–SMART DOMINATING FUNCTION.

$(\ell_{\max}, k)$–FOOLPROOF DOMINATING FUNCTION
INSTANCE: A graph H and a positive integer $j \leq |V(H)|$.
QUESTION: Does H have an $(\ell_{\max}, k)$–FDF of weight j or less?

This decision problem also belongs to the class NP, as shown below.

Proposition 4.16 *(ℓ, k)–FOOLPROOF DOMINATING FUNCTION $\in$ NP.*

Proof: The following algorithm outline is presented, to verify whether a specified guard function $f^{(0)}$ of a graph G is an (ℓ, k)–FDF of G with weight j or less.

Input: A graph G, guard function $f^{(0)}$ and an integer $j \leq |V(H)|$.
Step 1: Test whether $w(f^{(0)}) \leq j$ and $f^{(0)}(v) \leq \ell$ for every $v \in V(G)$. If true, continue. Otherwise return FALSE.
Step 2: For each problem sequence of length k, verify that any possible move strategy under $f^{(0)}$ can protect the problem sequence. If so, return TRUE. Otherwise return FALSE.

There are n possibilities for a problem vertex, where n is the order of G. No more than n move strategies exist that may (or may not) protect this problem vertex, resulting in a guard function $f^{(1)}$. The same is true for the next problem vertex under $f^{(1)}$. So, there are no more than n^{2k} possible move strategies to consider in Step 2. It follows that the algorithm will produce an output in polynomial time. ■

At the time of writing, a mapping from another NP–complete problem to (ℓ, k)–FOOLPROOF DOMINATING FUNCTION, necessary to prove the NP–completeness of this decision problem, has not yet been determined, although it is expected to exist.

4.6 Chapter Summary

In this chapter, a more general framework for the notion of finite order domination, as introduced by Burger *et al.* [2], was established. The two definitions, catering for smart and foolproof finite order domination was introduced in §4.1, generalising the definitions introduced by Burger *et al.* [2]. Various growth properties, concerning both an increase in the number of vertices requiring protection and the number of guards allowed per vertex, of these parameters were discussed in §4.2. In §4.3, the effect on the smart and foolproof parameters was examined when decomposing the graph structure by removing an edge. While the smart domination number behaves consistently, irrelevant of the graph in question or number of vertices requiring protection, further investigation is required to establish corresponding results for the foolproof parameter. An introduction to the issue of when to place multiple guards at a vertex was given in §4.4, by comparing the smart finite order domination number to the classical domination number, as originally conducted by Henning [18]. This section leads to arguably one of the more important issues on the topic of higher order domination, which may probably be clarified more easily through a comprehensive investigation of the finite order parameter values for special graph classes. By examining the parameter values of many different graph classes, a common characteristic of graphs for which multiple guards at a vertex does not decrease the parameter, may emerge. The chapter was concluded with a discussion on the complexity of the smart finite order parameter. It was proved by Henning [18] that the appropriate decision problem, for the case where no restriction is placed on the number of guards per vertex, is NP–complete. This result was discussed, and with it the expected difficulty in computing these parameters in general, confirmed.

Chapter 5

Infinite Higher Order Domination

In this chapter, definitions similar to those introduced by Burger *et al.* [3] (§3.5), are introduced in §5.1, providing for the protection of a graph against an infinite number of attacks. The existence of these parameters are confirmed in §5.2, and it is shown that there are only two infinite order parameters (§5.3). Finally, each of these two parameters are examined individually in §5.4 and §5.5 respectively.

5.1 Perpetual Graph Protection

Whereas the previous notions of dynamic graph protection, namely those in Definitions 4.1 and 4.2, required a specified, finite number of $k \in \mathbb{N}$ *problem vertices* to be protected by way of safe guard functions, the notion of protection may be extended to so–called perpetual security in a graph. The following two definitions, accommodating smart and foolproof infinite order domination respectively, allow for such an extension, generalising Definitions 3.10 and 3.11, as introduced by Burger *et al.* [2].

Definition 5.1 *A* **smart ∞–order ℓ–dominating function** *((ℓ, ∞)–SDF) is an (ℓ, k)–SDF in the limit as $k \to \infty$. The minimum weight of an (ℓ, ∞)–SDF for a graph G is denoted by*

$$\gamma_{\ell,\infty}(G) = \lim_{k\to\infty} \gamma_{\ell,k}(G),$$

which is called the **smart ∞–order ℓ–domination number** *of G.* ■

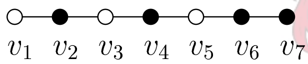
Definition 5.2 *A* **foolproof ∞–order ℓ–dominating function** *((ℓ, ∞)–FDF) is an (ℓ, k)–FDF in the limit as $k \to \infty$. The minimum weight of an (ℓ, ∞)–FDF for a graph G is denoted by*

$$\gamma^*_{\ell,\infty}(G) = \lim_{k\to\infty} \gamma^*_{\ell,k}(G)$$

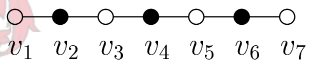
and is called the **foolproof ∞–order ℓ–domination number** *of G.* ■

These definitions are similar to their finite order counterparts, except that any number of problem vertices are now catered for. In the case of Definition 5.1 this means that if a safe guard function is an (ℓ, ∞)–SDF, then for any number of problem vertices, there exist corresponding moves from neighbouring guard–vertices so that a safe guard function results after every move. Referring to Definition 5.2, if a safe guard function is an (ℓ, ∞)–FDF, moves from *any* guard–vertices to a neighbouring unoccupied vertex will result in a safe guard function, after every move.

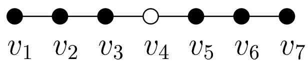
As an example, these concepts are illustrated in Figure 5.1 for a simple graph structure, namely the path P_7. The guard deployment configuration of a $(1, \infty)$–SDF for P_7 is shown in Figure 5.1(a) and it may be verified that, for any number of problem vertices, there exists a neigbouring guard–vertex to ensure that a safe guard function results. The guard on vertex v_2 may be used to move between v_1 and v_2 as needed, while the guard on vertex v_4 may be used to move between v_3 and v_4 as needed, etc. As will be shown later in this chapter, smart perpetual protection of P_7 cannot be achieved with fewer than 4 guards, so that $\gamma_{1,\infty}(P_7) = 4$. Figure 5.1(b) shows a safe guard configuration that is not a $(1, \infty)$–SDF, since there exists problem sequences of length 2, such as $\{v_5, v_3\}$, for which no guard–moves exist that result in a safe guard function after two moves. A guard configuration of a $(1, \infty)$–FDF for P_7 is shown in Figure 5.1(c), so that any number of guard–moves results in safe guard functions after each move, irrespective of which unoccupied and neighbouring occupied vertices are considered. The example shown in Figure 5.1(d), however, cannot protect P_7 against any infinite number of problem vertices. Figures 5.1(c)–(d) suggest that $\gamma^*_{1,\infty}(P_7) = 6$, which will be verified in the following sections.



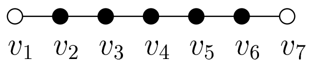
(a) *Example of a $(1, \infty)$–SDF configuration for P_7.*



(b) *Example of a safe guard function not being a $(1, \infty)$–SDF for P_7.*



(c) *Example of a $(1, \infty)$–FDF configuration for P_7.*



(d) *Example of a safe guard function not being a $(1, \infty)$–FDF for P_7.*

Figure 5.1: *Examples of guard configurations on P_7 for (a) a $(1, \infty)$–SDF and (c) a $(1, \infty)$–FDF. The configuration in (b) is an illustration of a safe guard function which is not a $(1, \infty)$–SDF. Similarly, a safe guard function that is not a $(1, \infty)$–FDF, is shown in (d).*

5.2 Existence of Parameters

The question of existence of the infinite order parameters, as defined in Definitions 5.1 and 5.2, is settled in general in the following theorem, extending the result by Burger *et al.* [3].

Theorem 5.1 *For any graph G of order n, the limits*

(a) $\gamma_{\ell,\infty}(G) = \lim_{k\to\infty} \gamma_{\ell,k}(G)$,

(b) $\gamma^*_{\ell,\infty}(G) = \lim_{k\to\infty} \gamma^*_{\ell,k}(G)$,

exist. In fact,

$$1 \leq \gamma_{\ell,\infty}(G) \leq \gamma^*_{\ell,\infty}(G) \leq n-1 \tag{5.1}$$

and both these bounds are attainable for both parameters by infinite classes of graphs.

Proof: The outermost inequalities in (5.1) trivially hold true if the parameters exist, and the existence of the limits follow from Proposition 4.3. The middle inequality in (5.1) may be proved by noting that, for a graph G, any (ℓ,∞)–FDF is also an (ℓ,∞)–SDF, the weights of which are bounded from below by $\gamma_{\ell,\infty}(G)$. The lower bound in (5.1) is attained when G is the complete graph K_n, while the upper bound is attained when G is the star $K_{1,n-1}$. ■

As expected, both the smart and foolproof infinite order parameters are upper bounds for their finite order counterparts respectively.

Proposition 5.1 *For any graph G and any $k \in \mathbb{N}$,*

(a) $\gamma_{\ell,k}(G) \leq \gamma_{\ell,\infty}(G)$, *for any* $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$,

(b) $\gamma^*_{\ell,k}(G) \leq \gamma^*_{\ell,\infty}(G)$, *for any* $\ell \in \{1, 2, \ldots, k+1\}$.

Proof: (a) Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ be an (ℓ,∞)–SDF of G with minimum weight $\gamma_{\ell,\infty}(G)$. Then, for any vertex sequence $v_0, v_1, \ldots$ of G, there exists a sequence $u_i \in N[v_i] \cap (V(G)\backslash V_0^{(i)})$, $i = 0, 1, \ldots$, that protects v_i, $i = 0, 1, \ldots$, under $f^{(0)}$. It follows that $f^{(0)}$ is also an (ℓ, k)–SDF of G, since this holds for any sequence $v_0, v_1, \ldots, v_{k-1}$ as well.

(b) Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ be an (ℓ,∞)–FDF of G with minimum weight $\gamma^*_{\ell,\infty}(G)$. Then, for any vertex sequence $v_0, v_1, \ldots$ of G, any sequence $u_i \in N[v_i] \cap (V(G)\backslash V_0^{(i)})$, $i = 0, 1, \ldots$, protects v_i, $i = 0, 1, \ldots$, under $f^{(0)}$. It follows that $f^{(0)}$ is also an (ℓ, k)–FDF of G, since this holds for any sequence $v_0, v_1, \ldots, v_{k-1}$ as well. ■

5.3 There are only Two Infinite Order Parameters

It is noted that the growth properties stated in Propositions 4.1 and 4.3 for the finite order parameters, still hold for the infinite order parameters. Also, the following growth relationships between the infinite order parameters with respect to increasing values of ℓ may be established, similar to the result of Proposition 4.2.

Proposition 5.2 *For any graph G with maximum degree Δ,*

(a) $\gamma_{\ell+1,\infty}(G) \leq \gamma_{\ell,\infty}(G)$ *for any* $\ell \in \{1, 2, \ldots, \Delta - 1\}$,

(b) $\gamma^*_{\ell+1,\infty}(G) \leq \gamma^*_{\ell,\infty}(G)$ *for any* $\ell \in \mathbb{N}$.

Proof: (a) By Definition 5.1, any (ℓ, ∞)–SDF of minimum weight is also an $(\ell+1, \infty)$–SDF, since the set $V^{(0)}_{\ell+1}$ of an $(\ell+1, \infty)$–SDF may be empty. The inequality follows, since the weights of all $(\ell + 1, \infty)$–SDFs of G are bounded from below by $\gamma_{\ell+1,\infty}(G)$.

(b) Similarly, by Definition 5.2, any (ℓ, ∞)–FDF of minimum weight is also an $(\ell+1, \infty)$–FDF, since the set $V^{(0)}_{\ell+1}$ of an $(\ell + 1, \infty)$–FDF may be empty. Because the weights of all $(\ell + 1, \infty)$–FDFs of G are bounded from below by $\gamma^*_{\ell+1,\infty}(G)$, the inequality follows. ■

Upon closer investigation of the smart and foolproof ∞–order secure and weak Roman domination parameters, Burger *et al.* [3] found that these domination numbers are equal, as mentioned in §3.5. As shown in the following two theorems, it is found that the smart and foolproof ∞–order parameters $\gamma_{\ell,\infty}(G)$ and $\gamma^*_{\ell,\infty}(G)$ are, in fact, also respectively equal for all $\ell \in \mathbb{N}$ and any graph G. The proofs of these theorems follow arguments similar to those in [3].

Theorem 5.2 *For any graph G with maximum degree Δ and any $\ell \in \{1, 2, \ldots, \Delta - 1\}$,*

$$\gamma_{\ell+1,\infty}(G) = \gamma_{\ell,\infty}(G).$$

Proof: By Proposition 5.2(a), $\gamma_{\ell+1,\infty}(G) \leq \gamma_{\ell,\infty}(G)$ for any $1 \leq \ell \leq \Delta - 1$. Suppose that there exists a graph G such that $\gamma_{\ell+1,\infty}(G) < \gamma_{\ell,\infty}(G)$ for some $\ell \in \mathbb{N}$ and let $f^{(0)} = (V^{(0)}_0, V^{(0)}_1, \ldots, V^{(0)}_{\ell+1})$ be an $(\ell + 1, \infty)$–SDF of G with minimum weight $w(f^{(0)}) = \gamma_{\ell+1,\infty}(G)$. Consider any set $S \subseteq V(G)$ and vertex sequence $v_i \in S$, $i = 0, 1, 2, \ldots$, with associated safe guard functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ and vertex sequence $u_i \in N[v_i] \cap (V(G)\backslash V^{(i)}_0)$, $i = 0, 1, 2 \ldots$ Because $\gamma_{\ell+1,\infty}(G) < \gamma_{\ell,\infty}(G)$, $f^{(k)}$ is not an (ℓ, ∞)–SDF for any $k \in \mathbb{N}_0$. This means that $V^{(k)}_{\ell+1} \neq \emptyset$ for all $k \in \mathbb{N}_0$. Hence there exists a vertex v^* for which $f^{(k)}(v^*) = \ell + 1$ for all $k \in \mathbb{N}_0$, despite the fact that $N(v^*) \subseteq S$ potentially. This means that the value of $f^{(0)}(v^*) \geq 2$ is not minimal, contradicting the fact that $f^{(0)}$ is a minimum weight $(\ell + 1, \infty)$–SDF. It is concluded that $\gamma_{\ell+1,\infty}(G) = \gamma_{\ell,\infty}(G)$ for any $1 \leq \ell \leq \Delta - 1$ and any graph G. ■

Theorem 5.3 *For any graph G and any $\ell \in \mathbb{N}$,*

$$\gamma^*_{\ell+1,\infty}(G) = \gamma^*_{\ell,\infty}(G).$$

Proof: It may be assumed that G is connected, since otherwise the upcoming proof argument may be applied to each component of G and the result will follow by utilisation of Lemma 4.2. By Proposition 5.2(b), $\gamma^*_{\ell+1,\infty}(G) \leq \gamma^*_{\ell,\infty}(G)$ for any $\ell \in \mathbb{N}$. Suppose that there exists a graph G such that $\gamma^*_{\ell+1,\infty}(G) < \gamma^*_{\ell,\infty}(G)$ for some $\ell \in \mathbb{N}$ and let $f^{(0)} = (V^{(0)}_0, V^{(0)}_1, \ldots, V^{(0)}_{\ell+1})$ be an $(\ell + 1, \infty)$–FDF of G with minimum weight

$w(f^{(0)}) = \gamma^*_{\ell+1,\infty}(G)$. Consider any sequence of vertices $\mathcal{S}_v = (v_0, v_1, v_2, \ldots)$ and any possible associated safe guard functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \rightarrow v_i)$ with vertex sequence $\mathcal{S}_u = (u_0, u_1, u_2, \ldots)$, $u_i \in N(v_i) \cap (V(G) \backslash V_0^{(i)})$, $i \in \mathbb{N}_0$. Because $\gamma^*_{\ell+1,\infty}(G) < \gamma^*_{\ell,\infty}(G)$, $f^{(k)}$ is not an (ℓ,∞)–FDF, for any $k \in \mathbb{N}_0$. This means that $V_{\ell+1}^{(k)} \neq \emptyset$, for all $k \in \mathbb{N}_0$. Hence there exists a vertex $v^* \in V_{\ell+1}^{(k)}$ for any $k \in \mathbb{N}_0$, such that v^* is not in $\mathcal{S}_u$. This can only hold if $N(v^*) \subseteq V(G) \backslash V_0^{(k)}$ for all $k \in \mathbb{N}_0$. Since $f^{(0)}$ is minimal, there exists a sequence of $k_1 \in \mathbb{N}_0$, say, unoccupied vertices, the protection of which results in a safe guard function $f^{(k_1)}$ such that $f^{(k_1)}(w) = 1$ for every $w \in N(v^*)$. Since $f^{(k)}(v^*) = \ell + 1$ for any $k \in \mathbb{N}_0$, $\{N(w) :\ w \in N(v^*)\} \subseteq V(G) \backslash V_0^{(k)}$ has to hold for all $k \geq k_1$. Repetition of this argument leads to the conclusion that $V(G) \subseteq V(G) \backslash V_0^{(k)}$ for all $k \geq k_2$, for some $k_2 \in \mathbb{N}_0$. Therefore $\gamma^*_{\ell+1,\infty}(G) = n$, contradicting the bounds in the existence Theorem 5.1. Hence $\gamma^*_{\ell+1,\infty}(G) = \gamma^*_{\ell,\infty}(G)$ for any ℓ and any graph G. ■

A fundamental difference in the proof strategies of Theorems 5.2 and 5.3 is noted. Though the proofs start out very similar, the smart and foolproof notions of higher order domination demand different strategies for proving essentially equivalent results.

From the results of Theorems 5.2 and 5.3, it follows that the ℓ–subscript is superfluous in the case of the infinite order parameters. Henceforth the parameter $\gamma_{\ell,\infty}(G)$ shall be denoted by $\gamma_\infty(G)$, and $\gamma^*_{\ell,\infty}(G)$ by $\gamma^*_\infty(G)$, similar to the notation used in [3]. In a similar vein, (ℓ,∞)–SDFs shall simply be referred to as ∞–smart dominating functions (∞–SDFs), while (ℓ,∞)–FDFs shall be referred to as ∞–foolproof dominating functions (∞–FDFs).

5.4 The Foolproof Parameter, γ^*_∞

Where Theorem 5.3 showed that the (ℓ,∞)–foolproof domination number for a graph G is the same, irrespective of $\ell \in \mathbb{N}$, resulting in a single parameter $\gamma^*_\infty(G)$, it is in fact possible to improve upon this result, by finding an exact value for this parameter explicitly. A similar argument to that followed in [3] may be applied to prove the following result.

Theorem 5.4 *For any connected order n graph G with minimal degree δ,*

$$\gamma^*_\infty(G) = n - \delta.$$

Proof: It suffices to prove the result for $\gamma^*_{1,\infty}(G)$, since $\gamma^*_{1,\infty}(G) = \gamma^*_\infty(G)$ by Theorem 5.3. Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ be any safe guard function of G, with $n - \delta \leq |V_1^{(0)}| \leq n$. If there exists a vertex sequence $v_0, v_1, \ldots, v_{k-1}$ for some $k \in \mathbb{N}$, for which a sequence $f^{(i+1)} = \text{move}(f^{(i)}, u_i \rightarrow v_i)$ result in $f^{(k)}$ not being a safe guard function, with $u_i \in N(v_i) \cap (V(G) \backslash V_0^{(i)})$ for all $i = 0, 1, \ldots, k-1$, then some vertex of G is left undominated. This means that $|V_0^{(0)}| = |V_0^{(k)}| \geq \delta + 1$, which is a contradiction, showing that $f^{(0)}$ is a $(1,\infty)$–FDF and that

$$\gamma^*_{1,\infty}(G) \leq n - \delta. \tag{5.2}$$

Suppose there existed a minimum weight $(1,\infty)$–FDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ such that $w(f^{(0)}) = \gamma^*_{1,\infty}(G) < n - \delta$, meaning that $V_0^{(0)} \geq \delta + 1$. For any given vertex sequence $v_0, v_1, v_2, \ldots$, any sequence of safe guard functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$, with $u_i \in N(v_i) \cap (V(G) \backslash V_0^{(i)})$ for all $i = 0, 1, 2, \ldots$, may follow. Therefore, for a vertex v^* in G of minimal degree, there exists a $k \in \mathbb{N}$ such that $N[v^*] \subseteq V_0^{(k)}$, resulting in v^* not being dominated. This contradiction shows that

$$\gamma^*_{1,\infty}(G) \geq n - \delta. \tag{5.3}$$

The desired result follows by combining inequalities (5.2) and (5.3). ■

If the graph G is disconnected, the value of the parameter $\gamma^*_\infty(G)$ can still be determined exactly, by using Lemma 5.1. The proof of this lemma is similar to that of Lemma 4.1.

Lemma 5.1 *For any disconnected graph G with components $H_1, H_2, \ldots, H_n$,*

$$\gamma^*_\infty(G) = \gamma^*_\infty(H_1) + \gamma^*_\infty(H_2) + \cdots + \gamma^*_\infty(H_n).$$

Proof: No two vertices in different components are connected. Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ be an ∞–FDF of G with minimum weight $w(f^{(0)}) = \gamma^*_\infty(G)$. Consider any component H of G. For any sequence of vertices $v_0, v_1, v_2, \ldots$ of H, any sequence $u_i \in N[v_i] \cap V_1^{(i)} \cap V(H)$, $i = 0, 1, \ldots$, protects v_i, $i = 0, 1, \ldots$, under $f^{(0)}$. For the component H, write $V_j^{(0)}(H) = V_j^{(0)} \cap V(H)$ for $j = 0, 1$. Then it follows that the safe guard function $g^{(0)} = (V_0^{(0)}(H), V_1^{(0)}(H))$ is an ∞–FDF for the component H. Therefore

$$\gamma^*_\infty(G) \geq \gamma^*_\infty(H_1) + \gamma^*_\infty(H_2) + \cdots + \gamma^*_\infty(H_n). \tag{5.4}$$

Let $g_t^{(0)} = (V_0^{(0)}(H_t), V_1^{(0)}(H_t))$ be a minimum weight ∞–FDF of the component H_t of G, $t = 1, 2, \ldots, n$. Also, let $\tilde{V}_j^{(0)} = V_j^{(0)}(H_1) \cup V_j^{(0)}(H_2) \cup \cdots \cup V_j^{(0)}(H_n)$ for $j = 0, 1$ and consider the safe guard function $\tilde{f}^{(0)} = (\tilde{V}_0^{(0)}, \tilde{V}_1^{(0)})$ of G. Then $w(\tilde{f}^{(0)}) = \gamma^*_\infty(H_1) + \cdots + \gamma^*_\infty(H_n)$, and for any sequence of vertices $v_0, v_1, v_2, \ldots$ of G, any sequence $u_i \in N[v_i] \cap \tilde{V}_1^{(i)}$, $i = 0, 1, 2, \ldots$ protects v_i, $i = 0, 1, \ldots$, under $f^{(0)}$, because $u_i \in N[v_i]$ only if $u_i, v_i \in H$ for some component H of G. Therefore it follows that

$$\gamma^*_\infty(G) \leq \gamma^*_\infty(H_1) + \gamma^*_\infty(H_2) + \cdots + \gamma^*_\infty(H_n). \tag{5.5}$$

By a combination of inequalities (5.4) and (5.5), the desired result follows. ■

5.5 The Smart Parameter, γ_∞

Finding the ∞–smart domination number, $\gamma_\infty(G)$, appears to be more difficult than is the case with its foolproof counterpart. As a first step in the investigation of this parameter, it may be shown that $\gamma_\infty(G)$ is bounded from above by the clique partition number $\mathfrak{c}(G)$, as investigated by Burger *et al.* [3].

Proposition 5.3 *If the vertex set of G may be partitioned into c subsets $S_1, \ldots, S_c$ such that S_j induces a clique in G for all $j = 1, \ldots, c$, then $\gamma_\infty(G) \leq c$.*

Proof: It suffices to show that $\gamma_{1,\infty}(G) \leq c$, since $\gamma_{\ell,\infty}(G) = \gamma_\infty(G)$ for any $1 \leq \ell \leq \Delta$, according to Theorem 5.2. Let $\mathcal{S} = \{S_1, \ldots, S_c\}$ be a partition of $V(G)$ such that, for any $S_j \in \mathcal{S}$, $\langle S_i \rangle$ is a clique, $j = 0, 1, \ldots, c$. Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G, with $V_1^{(0)} = \{w_j \in S_j : \ j = 1, 2, \ldots, c\}$ (i.e. a vertex from each subset $\mathcal{S}_i$) and $V_0^{(0)} = V(G) \backslash V_1^{(0)}$. Given any arbitrary vertex sequence $v_0, v_1, v_2, \ldots$, the sequence $u_i \in N(v_i) \cap V_1^{(i)} \cap \mathcal{S}_j$ if $v_i \in \mathcal{S}_j$, $j \in \{1, 2, \ldots, c\}$, protects v_i, $i = 0, 1, \ldots$, under $f^{(0)}$. Therefore $f^{(0)}$ is a $(1,\infty)$–SDF with weight $w(f^{(0)}) = |V_1^{(0)}| = c$, yielding the desired upper bound on $\gamma_\infty(G)$. ■

Determining the minimum value of c in Proposition 5.3 (i.e. $\mathfrak{c}(G)$) is, however, a known hard problem [26], called the minimum clique partition problem. Finding the clique partition number $\mathfrak{c}(G)$ of a graph G is equivalent to determining the vertex chromatic number $\chi(\overline{G})$ of the graph complement $\overline{G}$, as shown in Proposition 5.4. As mentioned in [3], no known m–optimal algorithm exists (m being a constant) for solving this problem. For example, the value of $\mathfrak{c}(G)$ is not necessarily obtained by selecting the partition with the largest clique in G, then choosing the next largest clique, and continuing in this fashion until all vertices are accommodated in some clique. Figure 5.2 (reproduced from [3]) is an example where this approach produces a value of $c = 4$, while in fact $\mathfrak{c}(G) = 3$. Further investigation as to when the clique partition number $\mathfrak{c}(G)$ (which is equal to $\chi(\overline{G})$) is in fact the same as the value of $\gamma_\infty(G)$, is clearly required.

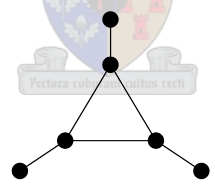


Figure 5.2: *This graph G may be partitioned into three subcliques of order two. (Reproduced from [3].)*

Proposition 5.4 *For any graph G, $\mathfrak{c}(G) = \chi(\overline{G})$.*

Proof: Let $S_1, \ldots, S_{\mathfrak{c}(G)}$ be a minimum clique partition of the graph G. Then S_i is an independent set in $\overline{G}$ for each $i = 1, \ldots, \mathfrak{c}(G)$, so that $\overline{G}$ may be coloured with $\mathfrak{c}(G)$ colours. Therefore, $\mathfrak{c}(G) \geq \chi(\overline{G})$.

Consider a minimal colouring of $\overline{G}$ and let the vertex subsets $S_1, \ldots, S_{\chi(\overline{G})}$ be a partition of $V(G)$ according to the colour classes of this colouring. Then S_i is independent in $\overline{G}$, and thus $\langle S_i \rangle$ is a clique in G for each $i = 1, \ldots, \chi(\overline{G})$. Therefore, $\mathfrak{c}(G) \leq \chi(\overline{G})$.

The result follows from these inequalities. ■

The above proposition implies that $\gamma_\infty(G) \leq \chi(\overline{G})$ for any graph G. As a first step, a lower bound on γ_∞ in terms of the independence number, β, may also be established by using Propositions 5.5 (proved in [3]) and Proposition 5.6.

Proposition 5.5 *For any graph G, $\gamma_\infty(G) \geq \beta(G)$.*

Proof: According to Theorem 5.2, it is sufficient to show that $\gamma_{1,\infty}(G) \geq \beta(G)$ for any graph G. Let $I = \{v_0, v_1, \ldots, v_{\beta(G)-1}\}$ be an independent vertex set of maximum cardinality in G. Suppose that $\gamma_{1,\infty}(G) < \beta(G)$, and let $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ be any minimum weight $(1,\infty)$–SDF of G. Consider the vertex sequence $v_0, v_1, \ldots, v_{k-1}$ for some $k \leq \beta(G)$. Then, irrespective of the vertex sequence $u_i \in N(v_i) \cap (V(G) \backslash V_0^{(i)})$, the sequence of guard functions $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ cannot result in a safe guard function $f^{(k)}$. This contradiction shows that $\gamma_{1,\infty}(G) \geq \beta(G)$, and thus that $\gamma_\infty(G) \geq \beta(G)$. ■

The following relationship between the independence number of a graph G and the clique number, $\omega(\overline{G})$, of the graph complement $\overline{G}$ may easily be confirmed.

Proposition 5.6 *For any graph G, $\beta(G) = \omega(\overline{G})$.*

Proof: Consider any graph G and let $\{v_1, v_2, \ldots, v_{\beta(G)}\} \subseteq V(G)$ be an independent set of G. This means that no v_i is adjacent to v_j, for $i, j = 1, 2, \ldots, \beta(G)$, and therefore $\langle v_1, v_2, \ldots, v_{\beta(G)} \rangle$ is a complete subgraph of $\overline{G}$. It follows that $\beta(G) \leq \omega(\overline{G})$.

Let $\langle v_1, v_2, \ldots, v_{\omega(\overline{G})} \rangle$ be a clique in $\overline{G}$. This means that the set $\{v_1, v_2, \ldots, v_{\omega(G)}\} \subseteq V(G)$ is independent in G, and therefore $\beta(G) \geq \omega(\overline{G})$.

The result follows from these inequalities. ■

It is now known that

$$\omega(\overline{G}) = \beta(G) \leq \gamma_\infty(G) \leq \mathfrak{c}(G) = \chi(\overline{G}) \tag{5.6}$$

for any graph G, as established by Burger *et al.* [3]. The upper bound is certainly sharp under certain circumstances, as is evident from the next proposition, obtained in [3].

Proposition 5.7 *If $\chi(\overline{G}) \leq 3$, then $\gamma_\infty(G) = \chi(\overline{G})$.*

Proof: If $\chi(\overline{G}) = 1$ the result trivially follows from inequality (5.6). If $\chi(\overline{G}) = 2$, there exist two vertices $u, v \in V(G)$ not adjacent in G (i.e. independent), meaning that $\gamma_\infty(G) \geq 2$. The result again follows from (5.6).

Let $\chi(\overline{G}) = 3$ and suppose that $\gamma_\infty(G) \leq 2$. Since $\gamma_\infty(G) \geq \omega(\overline{G})$, it follows that $\overline{G}$ is triangle–free. Suppose, however, that $\overline{G}$ does contain an odd cycle and denote the smallest odd cycle in $\overline{G}$ by $C : v_1 v_2 \cdots v_{2k+1}$ $(k \geq 2)$. Because v_1 and v_2 are independent in G, it follows that $\gamma_\infty(G) \geq 2$, and therefore $\gamma_\infty(G) = 2$. Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ be a $(1,\infty)$–SDF of G of minimum weight $w(f^{(0)}) = \gamma_\infty(G) = 2$ and consider the vertex sequence $v_1, v_2, v_4, v_6, \ldots, v_{2k}$. It follows that $v_1, v_2 \in V_1^{(2)}$. The safe guard functions $f^{(3)}, f^{(4)}, \ldots, f^{(k)}$ must emanate from the sequence $f^{(i+1)} = \text{move}(f^{(i)}, v_{2i-2} \to v_{2i})$, $i = 2, 3, \ldots, k-1$. This results in $f^{(k+1)}$ necessarily not being a safe guard function of G, since either v_{2k+1} or v_{2k-1} will be left undominated. It therefore follows that $\gamma_\infty(G) \neq 2$. This contradiction shows that $\overline{G}$ does not contain any odd cycles.

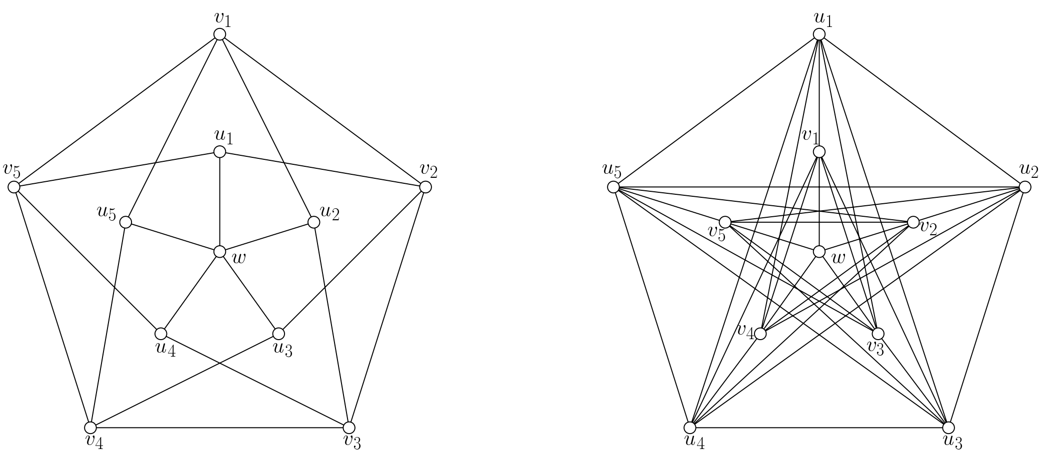
By Theorem 2.3 it follows that $\overline{G}$ is bipartite, implying that $\chi(\overline{G}) = 2$, which is a contradiction. It is concluded that $\gamma_\infty(G) > 2$ and therefore that $\gamma_\infty(G) = 3$ by utilisation of inequality (5.6). ■

Even though (5.6) holds for any graph G, it does not necessarily constitute tight bounds for the parameter $\gamma_\infty(G)$. The inequality $\omega(\overline{G}) \leq \chi(\overline{G})$ is sharp for complete graphs and some cycles, but it differs by one if $\overline{G}$ is an odd cycle of length more than 3, [5]. Furthermore, there exist families of graphs for which $\chi(\overline{G}) - \omega(\overline{G})$ can be made arbitrarily large. This follows from the next theorem, which was proved by Mycielski in 1955, among others [5]. Note that $\omega(G) \leq 2$ for any triangle–free graph G.

Theorem 5.5 *For every positive integer n, there exists an n-chromatic, triangle–free graph [5].* ■

Since $\chi(\overline{G}) - \omega(\overline{G})$ can be large, it raises the suspicion that a graph G may exist for which $\omega(\overline{G}) < \gamma_\infty(G) < \chi(\overline{G})$. An example of such a graph is found in the Grötzsch graph by way of the proof of Theorem 5.5. This order 11 graph, $\overline{G}$ say, is triangle–free and 4–chromatic, meaning that $\omega(\overline{G}) \leq 2$ and $\chi(\overline{G}) = 4$ [5]. Burger and Mynhardt [4] were able to prove that neither bound in (5.6) is sharp for the complement of the Grötzsch graph. Goddard *et al.* [12] were able to provide a similar result which applies to more graphs than just the Grötzsch graph. The proof of this result is provided here using notation consistent with the rest of the chapter.

Define $f = (V_0, V_1)$ to be a *safe guard function of type 2* of G if $f^{(0)}$ is a safe guard function of G, $\langle V_1 \rangle_G$ is not complete and $\beta(G) = 2$. The type designation corresponds to the independence number of the graph. This definition is employed in the proof of the following result.



(a) *The Grötzsch graph,* $\overline{G}$. (b) *The complement of the Grötzsch graph,* G.

Figure 5.3: *An example of a graph G for which $\omega(\overline{G}) < \gamma_\infty(G) < \chi(\overline{G})$.*

Proposition 5.8 *For any graph G, if $\beta(G) = 2$, then $\gamma_\infty(G) \le 3$.*

Proof: The result trivially holds if G is of order 3 or less, so let G be of order at least 4. Suppose $f^{(i)} = (V_0^{(i)}, V_1^{(i)})$ is a safe guard function of type 2 of G for some $i \in \mathbb{N}_0$ and $|V_1^{(i)}| = 3$. Let $V_1^{(i)}$ be the set $\{x, y, z\}$, with x and y independent in G. For any vertex $v_i \in V_1^{(i)}$, there exists a vertex $u_i \in V_1^{(i)}$ (the vertex $u_i = v_i$) such that $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ is a safe guard function of type 2 of G. Now consider any vertex $v_i \in V_0^{(i)}$. If v_i is adjacent to z, then the safe guard function $f^{(i+1)} = \text{move}(f^{(i)}, z \to v_i)$ is of type 2, by Proposition 2.1, since $\{x, y\} \subset V_1^{(i+1)}$ is a maximum independent set of G. If v_i is not adjacent to z, it must be adjacent to x and/or y by Proposition 2.1, since $\{x, y\}$ is a maximum independent set of G. Without loss of generality assume v_i is adjacent to x. Using Proposition 2.1 it follows that the safe guard function $f^{(i+1)} = \text{move}(f^{(i)}, x \to v_i)$ is a safe guard function of type 2 of G, since $\{z, v_i\} \subset V_1^{(i+1)}$ is a maximum independent set of G. It is concluded that, if the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ is of type 2 with $|V_1^{(0)}| = 3$, then, for any sequence of vertices $v_0, v_1, v_2, \ldots$, there exists a sequence of vertices $u_i \in V_1^{(i)}$ such that $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$ is a safe guard function of type 2 of G, for all $i = 0, 1, 2, \ldots$ Hence $f^{(0)}$ is an ∞–SDF of G and therefore $\gamma_\infty(G) \le 3$. ■

It may easily be verified that $\gamma_\infty(G) > 2$ if $\overline{G}$ is taken as the Grötzsch graph. Therefore it follows from Proposition 5.8 that $\gamma_\infty(G) = 3$. The following corollary summarises an important result discussed in this section, stating that there exist graphs G, for which the difference between $\gamma_\infty(G)$ and $\chi(\overline{G})$ can be made arbitrarily large.

Corollary 5.1 *For any $n \in \mathbb{N}$, there exists a graph G such that $\chi(\overline{G}) - \gamma_\infty(G) \ge n - 3$.*

Proof: From Theorem 5.5, for any $n \in \mathbb{N}$, there exists a triangle–free graph $\overline{G}$ such that $\chi(\overline{G}) = n$. It follows that for the triangle–free graph $\overline{G}$, $\beta(G) = \omega(\overline{G}) = 2$. Therefore $\gamma_\infty(G) \le 3$ from Proposition 5.8, yielding the desired result. ■

An open problem stated in [12] asked the question whether an $s_3 \in \mathbb{N}$ exists such that $\gamma_\infty(G) \le s_3 < \chi(\overline{G})$ for any graph G with $\beta(G) = 3$. For any graph G with $\beta(G) = 3$, it is intuitive to define a safe guard function $f = (V_0, V_1)$ of G to be of type 3 if V_1 contains a maximum independent set of G, and attempt to apply a similar argument to that of Proposition 5.8. The following observation provides evidence to suggest that such an approach may not yield the desired result.

Observation 5.1 *Define $f = (V_0, V_1)$ to be a* safe guard function of type 3 *of G if $\beta(G) = 3$ and V_1 contains a maximum independent set of G. For any $s_3 \ge 3$, $s_3 \in \mathbb{N}$, there exists a graph G such that $\beta(G) = 3$, $\chi(G) > s_3$, for which there exists a safe guard function $f^{(0)}$ of type 3 of G, with weight $w(f^{(0)}) = s_3$, that is not a $(1, 1)$–SDF of G.*

Proof: Let $m \in \mathbb{N}$ be such that $s_3 \le m$, and let G be the graph shown in Figure 5.4, where the vertices $\{x, y, z\}$ are each adjacent to every vertex in the subgraph isomorphic to K_m, as indicated.

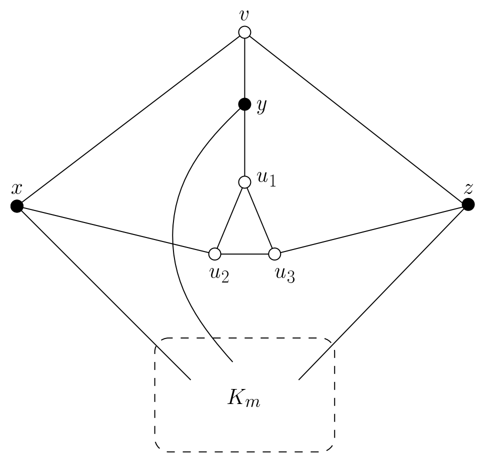


Figure 5.4: *The safe guard function (dominating set), consisting of the vertices $\{x, y, z\}$, as well as vertices in the K_m–subgraph, is a safe guard function of type 3. However, moving a guard to the vertex v does not produce a safe guard function.*

To show that $\chi(G) \geq m + 1$, note that the vertices of the K_m–subgraph of G, say $\{w_1, w_2, \ldots, w_m\}$ require m different colour classes. Since $\{x, y, z\}$ are each adjacent to every w_i, $i = 1, 2, \ldots, m$, it follows that $\chi(G) \geq m+1$. Partitioning $V(G)$ into the colour classes $\{u_1, v, w_1\}$, $\{u_2, w_2\}$, $\{u_3, w_3\}$, $\{x, y, z\}$ and $\{w_i\}$ for $i = 4, 5, \ldots, m$ if $m > 3$, shows that $\chi(G) \leq m + 1$. It follows that $\chi(G) = m + 1$ and therefore $s_3 < \chi(G)$.

Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$, with $V_1^{(0)} = \{x, y, z\}$ if $s_3 = 3$, and $V_1^{(0)} = \{x, y, z\} \cup \{w_1, w_2, \ldots, w_{s_3-3}\}$ if $s_3 > 3$. Clearly $w(f^{(0)}) = s_3$ and $f^{(0)}$ is a safe guard function of type 3 of G by Proposition 2.1, since $\{x, y, z\} \subseteq V_1^{(0)}$ is a maximum independent set of G. Considering the vertex v as indicated in Figure 5.4, it may easily be verified that $f^{(1)} = \text{move}(f^{(0)}, p \to v)$ is not a safe guard function for any $p \in \{x, y, z\}$. It follows that $f^{(0)}$ is not a $(1, 1)$–SDF of G. ∎

The open problem stated in [12] inquires about the existence of an integer $s_3 < \chi(\overline{G})$ such that, for any graph G with $\beta(G) \leq 3$, the ∞–smart domination number $\gamma_\infty(G) \leq s_3$. Suppose one attempts to prove this result for some $s_3 \geq 3$ using a similar argument to that of Proposition 5.8. Observation 5.1 shows that there exists a graph, of any order, for which some safe guard functions of type 3 may not appear in the move strategy of the ∞–SDF, since there exists a safe guard function of type 3 that cannot secure every problem vertex. It is concluded that a different approach is required to prove that $\gamma_\infty(G) \leq s_3 < \chi(\overline{G})$ for any graph G with $\beta(G) = 3$, since the existence of such an integer s_3 is still a possibility.

An immediate question arising from the inequality chain (5.6), namely $\beta(G) \leq \gamma_\infty(G) \leq \chi(\overline{G})$ for any graph G, is whether it is possible to characterise graphs for which either $\gamma_\infty(G) = \beta(G)$, $\gamma_\infty(G) = \chi(\overline{G})$, or both. The latter certainly holds if G is a perfect graph, although this condition is not necessary. Bipartite graphs, complete graphs, complete multipartite graphs and the cartesian product of two complete graphs are some examples of perfect graphs [12]. The problem still remains to determine for which graphs $\gamma_\infty(G) =$

$\beta(G) \neq \chi(\overline{G})$ or $\gamma_\infty(G) = \chi(\overline{G}) \neq \beta(G)$ holds true.

Much tighter bounds may exist for γ_∞ than those in (5.6) for some special graph classes. This will be examined further in Chapter 6. A useful result stated in [3], which follows immediately from Theorem 5.3, is stated in the following corollary.

Corollary 5.2 *If G is an order n graph such that, for some subset of vertices $W = \{v_1, \ldots, v_m\} \subseteq V(G)$, the graph $G - W$ possesses a perfect matching, then*

$$\gamma_\infty(G) \leq \frac{m+n}{2} = n - \nu(G).$$

Proof: If the graph $G - W$ possesses a perfect matching, then $\nu(G) = \mathfrak{c}(G-W) \leq \frac{n-m}{2}$. It follows that $\gamma_\infty(G) \leq \mathfrak{c}(G-W) + m \leq \frac{m+n}{2}$. ■

In conclusion it is noted that the inequality chain obtained by Burger *et al.* [3] and stated in Theorem 3.3, still holds true, in spite of the generalisation to allow for any number of guards per vertex.

Theorem 5.6 *The relationships*

$$\begin{array}{ccccccc}
\gamma(G) & \leq & \gamma_{\ell,k}(G) & \leq & \gamma_\infty(G) & \leq & \chi(\overline{G}) \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\gamma(G) & \leq & \gamma^*_{\ell,k}(G) & \leq & \gamma^*_\infty(G) & = & n - \delta
\end{array}$$

hold for all $k, \ell \in \mathbb{N}$ and any order n graph G with minimum degree δ. ■

5.6 Chapter Summary

In this chapter, the notion of protection of a graph was extended to allow for so–called perpetual security in a graph. Two new definitions, catering for smart and foolproof infinite order domination were provided in §5.1, generalising the definitions introduced by Burger *et al.* [3]. The existence of these infinite order parameters was confirmed in §5.2 and it was shown in §5.3 that there are, in fact, only two infinite order domination parameters. An exact value for the foolproof infinite order domination number was explicitly determined in §5.4, while its smart counterpart proved to be significantly more difficult to examine, as discussed in §5.5. General bounds on this parameter was obtained, though it is suspected that significantly tighter bounds may be found for some special graph classes.

Chapter 6

Special Graphs

Exact values of the parameters discussed in Chapters 4 and 5 may be established when considered for specific graph classes. In this chapter the special graph classes of complete graphs (§6.1), paths (§6.2), cycles (§6.3), products of the aforementioned graphs (§6.4), complete graphs (§6.5) and trees (§6.6) will be considered.

6.1 Complete Graphs

Determining the values of the parameters $\gamma_{\ell,k}$, $\gamma^*_{\ell,k}$, γ_∞ and γ^*_∞ for the complete graph K_n is simple and intuitive. The next two propositions provide these results.

Proposition 6.1 *For any $n, k \in \mathbb{N}$ and $\ell \in \{1, 2 \ldots, k+1\}$, $\gamma_{\ell,k}(K_n) = \gamma^*_{\ell,k}(K_n) = 1$.*

Proof: Consider any $v \in V(K_n)$. The safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$, with $V_1^{(0)} = \{v\}$ and $V_1^{(0)} = V(K_n)\backslash V_0^{(0)}$, is a $(1, k)$–FDF of K_n for any $k \in \mathbb{N}$, since for any sequence of k vertices $v_0, v_1, \ldots, v_{k-1}$, the sequence $u_0 = v$ and $u_i = v_{i-1}$ $(i = 1, 2, \ldots, k-1)$ protects v_i $(i = 1, 2, \ldots, k-1)$ under $f^{(0)}$. Therefore $\gamma_{\ell,k}(K_n) \leq \gamma^*_{\ell,k}(K_n) \leq \gamma^*_{1,k}(K_n) \leq w(f^{(0)}) = 1$. Since all the parameters have positive values, the desired result follows from Propositions 4.1 and 4.2. ∎

As expected, the values of the infinite order domination numbers follow as intuitively as the finite order parameters.

Proposition 6.2 *For any $n \in \mathbb{N}$, $\gamma_\infty(K_n) = \gamma^*_\infty(K_n) = 1$.*

Proof: The minimum vertex degree of the graph K_n is $\delta = n - 1$, so that $\gamma^*_\infty(K_n) = 1$ by Theorem 5.4. The desired result now follows from Theorem 5.1. ∎

6.2 Paths

In this section the various higher order domination parameters are determined for the path P_n. Since $\Delta = 2$ for any path P_n, $n \geq 3$, only $\ell \in \{1, 2\}$ needs to be considered for the smart parameter $\gamma_{\ell,k}(P_n)$. Hence the smart finite order domination numbers were completely determined by Burger *et al.* [2]. Using the arguments employed in [2] to prove Proposition 3.13(a), the values for the smart finite order parameters are provided in the following theorem. The only known result regarding the foolproof parameters is also reproduced from [2] below in Theorem 6.1(b).

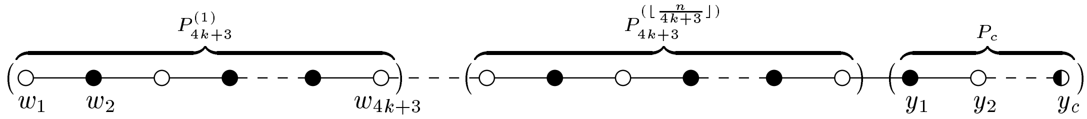
Theorem 6.1 *For any path P_n, $n \geq 3$,*

(a) $\gamma_{\ell,k}(P_n) = \left\lceil \dfrac{2k+1}{4k+3} n \right\rceil$, for any $k \in \mathbb{N}$ and $\ell \in \{1, 2 \ldots, \min(2, k+1)\}$.

(b) $\gamma^*_{1,k}(P_n) = \begin{cases} \lceil \frac{k+1}{k+3} n \rceil & \text{if } 1 \leq k \leq n-2 \\ n-1 & \text{if } k \geq n-1. \end{cases}$

Proof: (a) It is first shown that $\gamma_{1,k}(P_n) \leq \lceil \frac{2k+1}{4k+3} n \rceil$ for the path $P_n : w_1, \ldots, w_n$. Partition the path P_n into $\lfloor \frac{n}{4k+3} \rfloor$ subpaths $P^{(j)}_{4k+3}$ of order $4k+3$ ($j = 1, 2, \ldots, \lfloor \frac{n}{4k+3} \rfloor$) and one (possibly empty) subpath $P_c : y_1, \ldots, y_c$ of order $c \equiv n \pmod{4k+3}$, and consider the safe guard function $f^{(0)} = (V^{(0)}_0, V^{(0)}_1)$, with $V^{(0)}_1 = \{w_i : i \pmod{4k+3} \equiv 0 \pmod 2,\ w_i \in V(P^{(j)}_{4k+3}), j = 1, \ldots, \lfloor \frac{n}{4k+3} \rfloor\} \cup \{y_i \in V(P_c) : i \equiv 1 \pmod 2\}$ and $V^{(0)}_0 = V(P_n) \backslash V^{(0)}_1$. It will be shown that $f^{(0)}$ is a $(1, k)$–SDF of P_n, by showing that for any sequence of k vertices v_i (informally called problem vertices), there exists a sequence $u_i \in V^{(i)}_1 \cap N[v_i]$, $i = 0, \ldots, k-1$, that protects v_i, $i = 0, \ldots, k-1$, under $f^{(0)}$.

Pectora roborant cultus recti



Only the case where the whole sequence of problem vertices occurs in one subpath $P^{(t)}_{4k+3}$ needs to be considered, because if there exists such a vertex sequence u_i, $i = 0, 1, \ldots, k-1$, that renders safe guard functions $f^{(i)}$ for $P^{(t)}_{4k+3}$, $i = 0, \ldots, k$, given any sequence of k problem vertices v_i, $i = 0, 1, \ldots, k-1$, within the subpath, then there also exists a sequence u_i, $i = 0, \ldots, k-1$, that renders safe guard functions for fewer problem vertices within the subpath. Furthermore, since either P_c or $P_c - v$ possesses a perfect matching, it follows from Proposition 5.1 and Corollary 5.2 that $\gamma_{1,k}(P_c) \leq \gamma_\infty(P_c) \leq \lceil c/2 \rceil$ and hence for any sequence of k problem vertices $v_i \in V(P_c)$, $i = 0, 1, \ldots, k-1$, there exists a vertex sequence $u_i \in V(P_c)$ that protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$.

Therefore consider, without loss of generality, a sequence of k problem vertices $v_i \in V(P^{(1)}_{4k+3})$, $i = 0, 1, \ldots, k-1$. There are two possible cases:

<u>Case A:</u> $v_i \notin \{w_{4k+1}, w_{4k+2}, w_{4k+3}\}$ for any $i = 0, 1, \ldots, k-1$. In this case

$$\mathcal{M}_1 = \bigcup_{i=1}^{2k} \langle w_{2i-1}, w_{2i} \rangle$$

is a perfect matching of the subpath $\langle w_1, \ldots, w_{4k}\rangle$ of order $4k$. Therefore

$$\gamma_{1,k}(P^{(1)}_{4k+3}) \leq \gamma_\infty(\mathcal{M}_1) + \gamma(\langle w_{4k+1}, w_{4k+2}, w_{4k+3}\rangle) = 2k+1,$$

by utilisation of Corollary 5.2.

Case B: $v_i \in \{w_{4k+1}, w_{4k+2}, w_{4k+3}\}$ for some $i \in \{0, 1, \ldots, k-1\}$. In this case two further subcases may be distinguished:

Subcase B(i): $v_i \notin \{w_1, w_2, w_3\}$ for all $i = 0, 1, \ldots, k-1$. In this subcase

$$\mathcal{M}_2 = \bigcup_{i=2}^{2k+1} \langle w_{2i}, w_{2i+1}\rangle$$

is a perfect matching of the subpath $\langle w_4, \ldots, w_{4k+3}\rangle$ of order $4k$. It follows by a similar argument as in Case A, that $\gamma_{1,k}(P^{(1)}_{4k+3}) \leq 2k+1$.

Subcase B(ii): $v_i \in \{w_1, w_2, w_3\}$ for some $i \in \{0, 1, \ldots, k-1\}$. In this subcase there are at most $k-2$ problem vertices in the subpath $\langle w_4, \ldots, w_{4k}\rangle$. But then it follows, by the pigeonhole principle, that there are at least 4 consecutively labelled vertices that are not problem vertices: suppose they are $w_{2j}, w_{2j+1}, w_{2j+2}, w_{2j+3}$, for some $2 \leq j \leq 2k-2$ (the case where the first of these labels is odd, is similar). Then

$$\mathcal{M}_3 = \bigcup_{\alpha=1}^{j} \langle w_{2\alpha-1}, w_{2\alpha}\rangle \quad \text{and} \quad \mathcal{M}_4 = \bigcup_{\alpha=j+2}^{2k+1} \langle w_{2\alpha}, w_{2\alpha+1}\rangle$$

are perfect matchings of the subpaths $\langle w_1, \ldots, w_{2j}\rangle$ and $\langle w_{2j+4}, \ldots, w_{4k+3}\rangle$ of order $2k$ respectively, and it follows, by using Corollary 5.2, that

$$\gamma_{1,k}(P^{(1)}_{4k+3}) \leq \gamma_\infty(\mathcal{M}_3) + \gamma_\infty(\mathcal{M}_4) + \gamma(\langle w_{2j+1}, w_{2j+2}, w_{2j+3}\rangle) = 2k+1.$$

Consequently, in all cases it holds that

$$\gamma_{1,k}(P_n) \leq \left\lfloor \frac{n}{4k+3} \right\rfloor (2k+1) + \left\lceil \frac{c}{2} \right\rceil = \left\lceil \frac{2k+1}{4k+3} n \right\rceil. \tag{6.1}$$

The last equality may be proved by first showing that $\lceil \frac{c}{2} \rceil = \lceil \frac{2k+1}{4k+3} c \rceil$ if $c < 4k+3$. This process is performed in greater detail in Appendix A.3.

Suppose $f^{(0)} = (V^{(0)}_0, V^{(0)}_1, \ldots, V^{(0)}_\ell)$ is an (ℓ, k)–SDF of P_n, with weight $w(f^{(0)}) = \gamma_{\ell,k}(P_n) \leq \lceil \frac{2k+1}{4k+3} n \rceil - 1 = \lfloor \frac{n}{4k+3} \rfloor (2k+1) + \lceil \frac{c}{2} \rceil - 1$. Consider the subpath $P^{(1)}_{4k+3}$ and the set of (possible) problem vertices $I = \{w_{4m-1} : m = 1, \ldots, k\}$. Because this is an independent set, there exist vertices $u_i \in N[I] \cap (V(P^{(1)}_{4k+3}) \backslash V^{(i)}_0)$, $i = 0, 1, \ldots, k-1$, which protects I under $f^{(0)}$, so that $I \subseteq \cup_{j=1}^{\ell} V^{(k)}_j$. Furthermore, because $f^{(k)}$ must be a safe guard function, the set $J = \{w_{4m+1} : m = 0, 1, \ldots, k\}$ must also be dominated by vertices in $\cup_{j=1}^{\ell} V^{(k)}_j$. But no vertex in I is adjacent to vertices in J. Therefore $|(\cup_{j=1}^{\ell} V^{(k)}_j) \cap V(P^{(1)}_{4k+3})| \geq 2k+1$. Repeating this argument for all the subpaths $P^{(j)}_{4k+3}$, $j = 1, 2, \ldots, \lfloor \frac{n}{4k+3} \rfloor$, it follows that

$$\left| \left(\bigcup_{j=1}^{\ell} V^{(k)}_j \right) \cap \left(\bigcup_{j=1}^{\lfloor \frac{n}{4k+3} \rfloor} V(P^{(j)}_{4k+3}) \right) \right| \geq \left\lfloor \frac{n}{4k+3} \right\rfloor (2k+1).$$

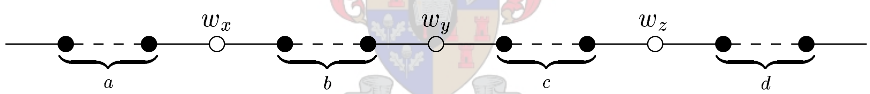
Therefore $P_c : y_1, \ldots, y_c$ contains at most $\lceil c/2 \rceil - 1$ vertices from $\cup_{j=1}^{\ell} V_j^{(0)}$, with the vertex y_1 necessarily protected by one of these vertices, following the above argument for the other subpaths. If $c \leq 2$, then a contradiction is obtained immediately. Therefore suppose $c \geq 3$. Considering the set of (possible) problem vertices $I = \{y_{4j-1} : j = 1, \ldots, \lfloor \frac{c+1}{4} \rfloor\}$, it again follows that there exist vertices $u_i \in N[I] \cap V(P_c) \backslash V_0^{(i)}$, $i = 0, 1, \ldots, k_1$, that protect I under $f^{(0)}$, so that $I \subseteq \cup_{j=1}^{\ell} V_j^{(k_1)}$, with $k_1 = \lfloor \frac{c+1}{4} \rfloor \leq k$. Furthermore, the set $J = \{y_{4j-3} : j = 1, 2, \ldots, \lceil \frac{c}{4} \rceil\}$ must also be dominated by vertices in $\cup_{j=1}^{\ell} V_j^{(k_1)}$. But no vertex in I is adjacent to vertices in J. Therefore

$$\left| \left(\bigcup_{j=1}^{\ell} V_j^{(k_1)} \right) \cap V\left(P_c\right) \right| \geq \left\lfloor \frac{c+1}{4} \right\rfloor + \left\lceil \frac{c}{4} \right\rceil > \left\lceil \frac{c}{2} \right\rceil - 1,$$

which is a contradiction.

It follows that $\gamma_{\ell,k}(P_n) \geq \lceil \frac{2k+1}{4k+3} n \rceil$ and thus $\gamma_{\ell,k}(P_n) = \lceil \frac{2k+1}{4k+3} n \rceil$, from (6.1) and Proposition 4.2.

(b) Consider the case $k \leq n-2$ and partition the path $P_n : w_1, \ldots, w_n$ into $\lfloor \frac{n}{k+3} \rfloor$ subpaths $P_{k+3}^{(j)} : w_{j(k+3)+1}, w_{j(k+3)+2}, \ldots, w_{j(k+3)+k+3}$, $j = 0, 1, \ldots, \lfloor \frac{n}{k+3} \rfloor - 1$ and one (possibly empty) subpath $P_c : w_{\lfloor n/(k+3) \rfloor (k+3)+1}, \ldots, w_n$ of order $c \equiv n \pmod{k+3}$. It is shown first, by contradiction, that every subpath $P_{k+3}^{(j)}$ contains at least $k+1$ vertices from $V_1^{(0)}$ for any $(1,k)$–FDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$. Suppose, to the contrary, that there exists a subpath $P_{k+3}^{(j)}$ of P_n containing only k vertices from $V_1^{(0)}$ (and hence 3 vertices from $V_0^{(0)}$). There is exactly one possible case:



Here dark vertices denote elements of $V_1^{(0)}$ and $a, b, c, d \geq 0$, with $a + b + c + d = k$. The sequences

$$f^{(j+1)} = \text{move}\left(f^{(j)}, w_{z-j-1} \rightarrow w_{z-j}\right), \quad j = 0, \ldots, c-1,$$

and

$$f^{(c+j+1)} = \text{move}(f^{(c+j)}, w_{x+j+1} \rightarrow w_{x+j}), \quad j = 0, \ldots, b-1,$$

render unsafe configurations in P_n after $b + c$ moves, because $w_y \in V_0^{(b+c)}$ is not adjacent to any other $w \in V_1^{(b+c)}$.

This contradiction shows that $|V(P_{k+3}^{(j)}) \cap V_1^{(0)}| \geq k+1$ for all $j = 0, 1, \ldots, \lfloor \frac{n}{k+3} \rfloor - 1$. In order to fulfil this property, it follows that $|V_1^{(0)}| \geq (k+1) \lfloor \frac{n}{k+3} \rfloor + r$, where $|V(P_c) \cap V_1^{(0)}| \geq r \geq \frac{k+1}{k+3} c$, rendering the lower bound

$$\gamma_{1,k}^*(P_n) \geq \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil \frac{k+1}{k+3} c \right\rceil, \quad k \leq n-2, \tag{6.2}$$

with $c \equiv n \pmod{k+3}$. To see that this bound is sharp, consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$, with

$$P_{k+3}^{(j)} \cap V_1^{(0)} = \{w_i \in V(P_{k+3}^{(j)}) : i \equiv 2, \ldots, k+2 \pmod{k+3}\}, \quad j = 0, \ldots, \left\lfloor \frac{n}{k+3} \right\rfloor - 1,$$

$$V(P_c)\cap V_1^{(0)} = \begin{cases} \{w_i \in V(P_c):\ i \equiv 1,2,\ldots,c \pmod{k+3}\} & \text{if } 1 \le c \le \lfloor (k+2)/2 \rfloor \\ \{w_i \in V(P_c):\ i \equiv 2,3,\ldots,c \pmod{k+3}\} & \text{if } \lfloor (k+2)/2 \rfloor < c < k+3, \end{cases}$$

and $V_0^{(0)} = V(P_n)\backslash V_1^{(0)}$. Clearly $f^{(0)}$ is a $(1,k)$–FDF for P_n, and hence

$$\begin{aligned} \gamma^*_{1,k}(P_n) &\le w(f^{(0)}) \\ &= \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + (c+1) - \left\lceil \frac{2c+1}{k+3} \right\rceil \\ &\le \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil \frac{k+1}{k+3} c \right\rceil . \end{aligned} \tag{6.3}$$

Appendix A.4 shows in greater detail how the final inequality in (6.3) is obtained. It follows by (6.2) and (6.3) that

$$\gamma^*_{1,k}(P_n) = \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil \frac{k+1}{k+3} c \right\rceil = \left\lceil \frac{k+1}{k+3} n \right\rceil, \qquad k \le n-2.$$

The last equality may be proved using the identity $\lceil a+b \rceil = a + \lceil b \rceil$ for all $a \in \mathbb{Z}$ and $b \in \mathbb{R}$ (see Proposition A.2 in Appendix A.2).

Finally note that

$$\left\lceil \frac{k+1}{k+3} n \right\rceil = n-1 \quad \text{if} \quad k = n-2.$$

It follows by Proposition 4.3(b) that

$$\gamma^*_{1,k}(P_n) \ge \gamma^*_{1,n-2}(P_n) = n-1 \tag{6.4}$$

for any $k \ge n-2$. But, by Theorem 5.4, we have $\gamma^*_\infty(P_n) = n-1$. Therefore it follows, again via Proposition 4.3(b) that

$$\gamma^*_{1,k}(P_n) \le \gamma^*_\infty(P_n) = n-1 \tag{6.5}$$

for all $k \in \mathbb{N}$. A combination of (6.4) and (6.5) yields the desired result, namely that $\gamma^*_{1,k}(P_n) = n-1$ for all $k \ge n-2$. ■

As stated by Burger *et al.* [2], the expectation exists that $\gamma^*_{\ell,k}(P_n) = \gamma^*_{1,k}(P_n)$ for any $\ell \in \{1, 2\ldots, k+1\}$, although this is yet to be confirmed. Using an argument similar to that in the proof of Theorem 6.1(b), the result does hold for the case $k=1$, however, as shown below.

Proposition 6.3 *For any path P_n, $\gamma^*_{\ell,1}(P_n) = \gamma^*_{1,1}(P_n) = \lceil \frac{n}{2} \rceil$ for any $\ell \in \{1, 2\ldots, k+1\}$.*

Proof: Let $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, V_2^{(0)})$ be a $(2,1)$–FDF of P_n with minimum weight $w(f^{(0)}) = \gamma_{2,1}(P_n)$. Partition P_n into $\lfloor \frac{n}{4} \rfloor$ subpaths $P_4^{(j)}$, $j = 1, 2, \ldots, \lfloor \frac{n}{4} \rfloor$, and one (possibly empty) subpath P_c, $0 \leq c \leq 3$. Then clearly $|V(P_4^{(j)}) \cap (V_1^{(0)} \cup V_2^{(0)})| \geq 2$, $j = 1, 2, \ldots, \lfloor \frac{n}{4} \rfloor$. Therefore, with $k = 1$,

$$\begin{aligned}\gamma^*_{2,1}(P_n) &\geq \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil \frac{k+1}{k+3} c \right\rceil \\ &= \left\lceil \frac{k+1}{k+3} n \right\rceil \\ &= \gamma^*_{1,1}(P_n),\end{aligned}$$

by the same argument as in the proof of Theorem 6.1(b). Since $\gamma^*_{\ell,1}(P_n) = \gamma^*_{2,1}(P_n)$ for all $\ell \geq 2$ by Proposition 4.4(b), it follows that $\gamma^*_{\ell,1}(P_n) \geq \gamma^*_{1,1}(P_n)$ for any $\ell \in \mathbb{N}$.

From Proposition 4.2(b) it is also known that $\gamma^*_{\ell,1}(P_n) \leq \gamma^*_{1,1}(P_n)$ for any $\ell \in \mathbb{N}$, so that the desired result follows. ■

By utilising Theorem 6.1(a), the value for the infinite order domination numbers may be obtained easily, as was established by Burger *et al.* [3]. This is stated in the following corollary.

Corollary 6.1 *For any path P_n,*

(a) $\gamma_\infty(P_n) = \left\lceil \frac{n}{2} \right\rceil$,

(b) $\gamma^*_\infty(P_n) = n - 1$.

Proof: (a) Since $\lim_{k\to\infty} \frac{2k+1}{4k+3} = \frac{1}{2}$, it follows from Definition 5.1 that $\gamma_\infty(P_n) = \lim_{k\to\infty} \gamma_{1,k}(P_n) = \lceil \frac{n}{2} \rceil$.

(b) From Theorem 5.4 it immediately follows that $\gamma^*_\infty(P_n) = n - 1$, since the minimum vertex degree of P_n is $\delta = 1$. ■

6.3 Cycles and Wheels

Since $\Delta = 2$ for any cycle C_n, only $\ell \in \{1, 2\}$ needs to be considered, so that the smart finite order domination numbers were complete determined by Burger *et al.* [2], as stated in Proposition 3.14(a). This result was achieved through arguments similar to those in the proof of Thereom 6.1 and is provided in the following theorem. Similar to the case for paths, only the $(1, k)$–foolpoof domination number was found for cycles.

Theorem 6.2 *For any cycle C_n, $n \geq 4$,*

(a) $\gamma_{\ell,k}(C_n) = \left\lceil \frac{2k+1}{4k+3} n \right\rceil$, *for any $k \in \mathbb{N}$ and $\ell \in \{1, 2 \ldots, \min(2, k+1)\}$.*

(b) $\gamma^*_{1,k}(C_n) = \begin{cases} \left\lceil \frac{k+1}{k+3} n \right\rceil, & \text{if } 1 \leq k \leq n-3 \\ n-2, & \text{if } k \geq n-3. \end{cases}$

Proof: (a) Since P_n is a spanning subgraph of C_n, it immediately follows from Corollary 4.2 that

$$\gamma_{\ell,k}(C_n) \leq \gamma_{\ell,k}(P_n) = \left\lceil \frac{2k+1}{4k+3}n \right\rceil. \tag{6.6}$$

To prove that $\gamma_{\ell,k}(C_n) \geq \lceil \frac{2k+1}{4k+3}n \rceil$, two cases are distinguished. Firstly, suppose that $n \geq 4k+3$ and that $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ,k)–SDF of C_n with minimum weight $w(f^{(0)}) = \gamma_{\ell,k}(C_n) < \lceil \frac{2k+1}{4k+3}n \rceil$. Then $w(f^{(0)}) < \frac{2k+1}{4k+3}n$. It follows that there exists a subpath $W \cong P_{4k+3}$ of C_n with $f^{(0)}(V(W)) < 2k+1$, since otherwise, if $f^{(0)}(V(W)) \geq 2k+1$ for every subpath $W \cong P_{4k+3}$ of C_n, then $w(f^{(0)}) \geq \frac{2k+1}{4k+3}n$. Denote this subpath by $W = \langle w_1, w_2, \ldots, w_{4k+3} \rangle$. Let $I = \{w_{4j} : j = 1, 2, \ldots, k\}$ and $J = \{w_{4j+2} : j = 0, 1, \ldots, k\}$. Since I is an independent set, there exist vertices $u_i \in N[I] \cap (V(W) \backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$, that protect the vertex sequence w_{4i}, $i = 1, 2, \ldots, k$, under $f^{(0)}$, so that $I \subseteq \cup_{j=1}^{\ell} V_j^{(k)}$. Because $f^{(k)}$ is a safe guard function of C_n, the set J must also be dominated by vertices in $\cup_{j=1}^{\ell} V_j^{(k)}$. But no vertex in I is adjacent to vertices in J. Therefore it follows that $f^{(0)}(V(W)) \geq 2k+1$, which is a contradiction. It follows that $\gamma_{\ell,k}(C_n) \geq \lceil \frac{2k+1}{4k+3}n \rceil$.

Now suppose that $n < 4k+3$ and that $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ,k)–SDF of C_n with minimum weight $w(f^{(0)}) = \gamma_{\ell,k}(C_n) < \lceil \frac{2k+1}{4k+3}n \rceil = \lceil \frac{n}{2} \rceil$ by utilisation of the argument in Appendix A.3. Then $w(f^{(0)}) < \frac{n}{2}$. Denote the cycle by $C_n : w_1, w_2, \ldots, w_n$. It follows that there exist adjacent vertices $x, y \in V(C_n)$ with $f^{(0)}(\{x,y\}) = 0$, because if $f^{(0)}(\{x,y\}) \geq 1$ for every adjacent pair $x, y \in V(C_n)$, then $w(f^{(0)}) \geq \frac{n}{2}$. Without loss of generality, assume $f^{(0)}(\{w_{n-2}, w_{n-1}\}) = 0$. Also assume that $n > 4$, since obtaining a contradiction for the case $n \leq 3$ is a trivial matter. It follows that $f^{(0)}(w_{n-3}) = f^{(0)}(w_n) = 1$, since $f^{(0)}$ is a safe guard function of C_n. Partition the cycle C_n into two subpaths, $\langle w_{n-3}, w_{n-2}, w_{n-1}, w_n \rangle$ and $P_c : w_1, \ldots, w_{n-4}$, $c = n-4 \leq 4k-2$. Note that the vertices w_{n-4} and w_{n-1} cannot be used to protect any vertex sequence $v_i \in V(P_c)$, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$. Let

$$I = \begin{cases} \{w_{4j-3} : j = 1, 2, \ldots, \lceil \frac{c}{4} \rceil - 1\} \cup \{w_c\} & \text{if } c \equiv 3 \pmod 4 \\ \{w_{4j-3} : j = 1, 2, \ldots, \lceil \frac{c}{4} \rceil\} & \text{otherwise} \end{cases}$$

and

$$J = \begin{cases} \{w_{4j-1} : j = 1, 2, \ldots, \lfloor \frac{c+1}{4} \rfloor - 1\} \cup \{w_{c-2}\} & \text{if } c \equiv 3 \pmod 4 \\ \{w_{4j-1} : j = 1, 2, \ldots, \lfloor \frac{c+1}{4} \rfloor\} & \text{otherwise.} \end{cases}$$

Since I is an independent set, it holds that for any vertex sequence v_i in I, $i = 0, 1, \ldots, k-1$, there exists a sequence $u_i \in N[I] \cap (V(P_c) \backslash V_0^{(i)})$, $i = 0, 1, \ldots, k-1$, that protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$, so that $I \subseteq \cup_{j=1}^{\ell} V_j^{(k_1)}$, with $k_1 = \lceil \frac{c}{4} \rceil \leq k$. Furthermore, since $f^{(k_1)}$ is a safe guard function of C_n, the set J must also be dominated by vertices in $\cup_{j=1}^{\ell} V_j^{(k_1)}$. But $N(\{w_{n-3}, w_n\}) \cap J = \emptyset$ and no vertex in I is adjacent to vertices in J. Therefore it holds that $f^{(0)}(V(P_c)) \geq \lfloor \frac{c+1}{4} \rfloor + \lceil \frac{c}{4} \rceil = \lceil \frac{c}{2} \rceil$, and hence $f^{(0)}(V(C_n)) \geq \lceil \frac{c}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil = \lceil \frac{2k+1}{4k+3}n \rceil$ by utilisation of Proposition A.2 in Appendix A.2 and the argument in Appendix A.3.

In both cases it holds that

$$\gamma_{\ell,k}(C_n) \geq \left\lceil \frac{2k+1}{4k+3}n \right\rceil, \tag{6.7}$$

so that the desired result follows by a combination of (6.6) and (6.7).

(b) It can be shown, by exactly the same argument as in Theorem 6.1(b), that

$$\gamma^*_{1,k}(C_n) \geq \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil \frac{k+1}{k+3} c \right\rceil = \left\lceil \frac{k+1}{k+3} n \right\rceil, \tag{6.8}$$

with $c \equiv n \pmod{k+3}$. To see that this bound is sharp, partition the cycle C_n into $\lfloor \frac{n}{k+3} \rfloor$ subpaths $P^{(j)}_{k+3} : v_{j(k+3)+1}, v_{j(k+3)+2}, \ldots, v_{j(k+3)+k+3}$, $j = 0, 1, \ldots, \lfloor \frac{n}{k+3} \rfloor - 1$ and one (possibly empty) subpath $P_c : v_{\lfloor n/(k+3) \rfloor (k+3)+1}, \ldots, v_n$ of order $c \equiv n \pmod{k+3}$. Consider the safe guard function $f^{(0)} = (V^{(0)}_0, V^{(0)}_1)$, with

$$V(P^{(j)}_{k+3}) \cap V^{(0)}_0 = \left\{ v_i \in V(P^{(j)}_{k+3}) : \ i \equiv \left\lfloor \frac{k+2}{2} \right\rfloor, k+2 \pmod{k+3} \right\}$$

for $j = 0, 1, \ldots, \lfloor \frac{n}{k+3} \rfloor - 1$, with

$$V(P_c) \cap V^{(0)}_0 = \left\{ v_i \in V(P_c) : \ i \equiv \left\lfloor \frac{k+2}{2} \right\rfloor \pmod{k+3} \right\}$$

if $c > \lfloor \frac{k+2}{2} \rfloor$, and with $V^{(0)}_1 = V(C_n) \backslash V^{(0)}_0$. Clearly $f^{(0)}$ is a $(1,k)$–FDF of C_n, since any vertex sequence of length $i \leq \lfloor \frac{k+2}{2} \rfloor + (k+2) - 1 - \lfloor \frac{k+2}{2} \rfloor = k+1$ can be protected under $f^{(0)}$. Hence

$$\begin{aligned} \gamma^*_{1,k}(C_n) \ \leq \ w(f^{(0)}) \ &\leq \ \begin{cases} \lfloor \frac{n}{k+3} \rfloor (k+1) + c, & \text{if } 1 \leq c \leq \lfloor \frac{k+2}{2} \rfloor \\ \lfloor \frac{n}{k+3} \rfloor (k+1) + c - 1, & \text{if } \lfloor \frac{k+2}{2} \rfloor < c < k+3 \end{cases} \\ &\leq \ \left\lceil \frac{k+1}{k+3} n \right\rceil \end{aligned} \tag{6.9}$$

for all $k \leq n-3$, exactly as in the proof of Theorem 6.1(b). The desired result for $k \leq n-3$ therefore follows by a combination of (6.8) and (6.9).

Finally note that

$$\left\lceil \frac{k+1}{k+3} n \right\rceil = n - 2 \quad \text{if} \quad k = n-3.$$

It follows by Proposition 4.3(b) that

$$\gamma^*_{1,k}(C_n) \geq \gamma^*_{1,n-3}(C_n) = n-2 \tag{6.10}$$

for any $k \geq n-3$. But certainly

$$\gamma^*_{1,k}(C_n) \leq n-2 \tag{6.11}$$

for all $k \in \mathbb{N}$, from Proposition 5.1 and Theorem 5.4. A combination of (6.10) and (6.11) yields the desired result, namely that $\gamma^*_{1,k}(C_n) = n-2$ for all $k \geq n-3$. ■

The values of the infinite order domination numbers of cycles may again be obtained easily, as was shown by Burger *et al.* [3]. This is stated in the following corollary.

Corollary 6.2 *For any cycle C_n, $n \geq 3$,*

(a) $\gamma_\infty(C_n) = \left\lceil \frac{n}{2} \right\rceil$,

(b) $\gamma^*_\infty(C_n) = n - 2$.

Proof: (a) Since $\lim_{k\to\infty} \frac{2k+1}{4k+3} = \frac{1}{2}$, it follows from Definition 5.1 that $\gamma_\infty(C_n) = \lim_{k\to\infty} \gamma_{1,k}(P_n) = \lceil \frac{n}{2} \rceil$.

(b) From Theorem 5.4 it immediately follows that $\gamma^*_\infty(C_n) = n - 2$, since the minimum vertex degree of C_n is $\delta = 2$. ■

From the proof of Theorem 6.2(a) it is noted that, if $n < 4k+3$, then $\gamma_{\ell,k}(C_n) = \gamma_\infty(C_n)$ for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$. The question of obtaining a k' such that $\gamma_{\ell,k'}(C_n) = \gamma_\infty(C_n)$ for all $k \geq k'$, follows from this observation. It is clear that the value of $\gamma_{\ell,k}(C_n) = \lceil \frac{2k+1}{4k+3} n \rceil$ increases as k increases for a fixed value of n, and also that $\gamma_{\ell,k'}(C_n) = \gamma_\infty(C_n)$ if and only if $\frac{2k+1}{4k+3} n > \lceil \frac{n}{2} \rceil - 1$. From this it follows that $n < 4k' - 3$ if n is odd, while $n < 8k' - 6$ if n is even, so that

$$k' = \begin{cases} \lceil \frac{n-6}{8} \rceil & \text{if } n \text{ is even} \\ \lceil \frac{2n-6}{8} \rceil & \text{if } n \text{ is odd.} \end{cases}$$

A question following from the above mentioned observation, is whether values for k' may be determined for other graph classes.

A special class of graphs related to both cycles and stars, is the class of wheels W_n. An upper bound on the smart finite order domination number of this graph may be obtained and is, in fact, expected to be the exact value.

Pectora roborant cultus recti

Proposition 6.4 *For the wheel W_n,*

$$\gamma_{\ell,k}(W_n) \leq \left\lceil \frac{2k+1}{4k+3} n \right\rceil$$

for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \ell_{\max}\}$, with $\ell_{\max} = \min(n, k+1)$.

Proof: Denote the vertices of W_n by $S = \{w_0, w_1, \ldots, w_{n-1}\}$ and w, such that $\langle S \rangle \cong C_n$ and w is the hub of W_n. Consider a safe guard function $f^{(0)}$ of W_n with weight $w(f^{(0)}) = \gamma_{\ell,k}(C_n) = \lceil \frac{2k+1}{4k+3} n \rceil$, such that $f^{(0)}(w) = 0$. Since $f^{(0)}$ is an (ℓ, k)–SDF of C_n and $\langle w, w_j, w_{j+1 \pmod n} \rangle \cong K_3$ for every $j \in \{0, 1, \ldots, n-1\}$, it follows that $f^{(0)}$ is also an (ℓ, k)–SDF of W_n, with weight $w(f^{(0)}) = \lceil \frac{2k+1}{4k+3} n \rceil$. ■

In the case where no restriction is placed on the number of guards per vertex, the smart finite order parameter value may be even less.

Proposition 6.5 *For the wheel W_n,*

$$\gamma_{\ell_{\max},k}(W_n) \leq \min\left(k+1, \left\lceil \frac{2k+1}{4k+3} n \right\rceil\right),$$

for any $k \in \mathbb{N}$, with $\ell_{\max} = \min(n, k+1)$.

Proof: Denote the vertices of W_n by $S = \{w_0, w_1, \ldots, w_{n-1}\}$ and w, such that $\langle S \rangle \cong C_n$ and w is the hub of W_n. By Proposition 6.4, it holds that $\gamma_{\ell,k}(W_n) \leq \lceil \frac{2k+1}{4k+3} n \rceil$. Suppose $\lceil \frac{2k+1}{4k+3} n \rceil \geq k+1$. Since $W_n - \{w_j w_{j+1 \pmod n} : j = 0, 1, \ldots, n-1\} \cong K_{1,n}$ is a spanning subgraph of W_n, it follows from Corollary 4.2 and Proposition 4.11 that $\gamma_{\ell_{\max},k}(W_n) \leq \gamma_{\ell_{\max},k}(K_{1,n}) \leq k+1$, since $\Delta = n \geq \lceil \frac{2k+1}{4k+3} n \rceil \geq k+1$. ■

It is expected that the upperbounds provided in these two propositions are sharp. In the case of the smart infinite order domination number of the wheel the parameter value may, however, be determined exactly.

Proposition 6.6 *For the wheel W_n, $\gamma_\infty(W_n) = \lceil \frac{n}{2} \rceil$.*

Proof: Denote the vertices of W_n by $S = \{w_0, w_1, \ldots, w_{n-1}\}$ and w, such that $\langle S \rangle \cong C_n$ and w is the hub of W_n. Consider a safe guard function $f^{(0)}$ of W_n with weight $w(f^{(0)}) = \gamma_\infty(C_n) = \lceil \frac{n}{2} \rceil$, such that $f^{(0)}(w) = 0$. Since $f^{(0)}$ is an ∞–SDF of C_n and $\langle w, w_j, w_{j+1 \pmod n} \rangle \cong K_3$ for every $j \in \{0, 1, \ldots, n-1\}$, it follows that $f^{(0)}$ is also an ∞–SDF of W_n, so that $\gamma_\infty(W_n) \leq \lceil \frac{n}{2} \rceil$.

Suppose $f^{(0)}$ is an ∞–SDF of W_n with weight $w(f^{(0)}) = \gamma_\infty(W_n) < \lceil \frac{n}{2} \rceil$. Then there exists a vertex sequence $v_0, v_1, \ldots, v_{c-1}$ of length $c \in \mathbb{N}$ in S, such that its protection results in a safe guard function $f^{(c)}$, with $f^{(c)}(w) = 0$. Since $\langle S \rangle \cong C_n$, it follows that any vertex sequence $v_c, v_{c+1}, \ldots$ in S is necessarily protected by some vertex sequence also in S. Therefore $f^{(c)}$ is an ∞–SDF of C_n of weight $w(f^{(c)}) < \lceil \frac{n}{2} \rceil$, which contradicts Corollary 6.2. It is concluded that $\gamma_\infty(W_n) \geq \lceil \frac{n}{2} \rceil$, providing the desired result. ■

Although the value of the foolproof finite order domination number of a wheel is still undetermined, the infinite–order parameter value follows immediately from Theorem 5.4, and is stated below.

Corollary 6.3 *For the wheel W_n, $\gamma^*_\infty(W_n) = n - 2$.*

Proof: Since the minimum degree of W_n is $\delta = 3$ and the order of the graph is $n+1$, the result follows directly from Theorem 5.4. ■

6.4 Products of Complete Graphs, Paths and Cycles

In this section, various cartesian products of graphs are considered, as initially discussed by Burger *et al.* [3]. The results contained in Proposition 3.18 are discussed, following the proof arguments used in [3]. Furthermore, the lower bound in Proposition 3.18(e) is improved. The following preliminary result concerning the independence number is useful.

Proposition 6.7 *For the complete graphs K_p and K_q, $p \leq q$, $\beta(K_p \times K_q) = p$.*

Proof: Denote the set $V(K_p \times K_q)$ by $\cup_{i=1}^{p}\{s_{i,1}, s_{i,2}, \ldots, s_{i,q}\}$, with $\langle s_{i,1}, s_{i,2}, \ldots, s_{i,q}\rangle \cong K_q$ for $i = 1, 2, \ldots, p$. Then it follows by the definition of a cartesian product that the set $\{s_{i,i} : i = 1, 2, \ldots, p\}$ is an independent set of $K_p \times K_q$, and hence $\beta(K_p \times K_q) \geq p$.

Suppose $\beta(K_p \times K_q) > p$ and consider an independent set $\{v_1, v_2, \ldots, v_\beta\}$ of $K_p \times K_q$. Then, by way of the pigeonhole principle, it follows that $\{v_{j_1}, v_{j_2}\} \subseteq \{s_{i,1}, s_{i,2}, \ldots, s_{i,q}\}$ for some $j_1, j_2 \in \{1, 2, \ldots, \beta\}$ and some $i \in \{1, 2, \ldots, p\}$. Since v_{j_1} and v_{j_2} are necessarily adjacent, a contradiction is obtained. Hence $\beta(K_p \times K_q) \leq p$, and the result follows by a combination with the previous inequality. ■

Utilising the above result, in combination with Proposition 5.5, the following result regarding the infinite order domination numbers, was obtained by Burger *et al.* [3].

Proposition 6.8 *For the complete graphs K_p and K_q, $p \leq q$,*

(a) $\gamma_\infty(K_p \times K_q) = p$,

(b) $\gamma^*_\infty(K_p \times K_q) = pq - p - q + 2$.

Proof: (a) The set $V(K_p \times K_q)$ may be partitioned into p subsets, each inducing a clique of order q in $K_p \times K_q$. Hence it follows from Proposition 5.3 that $\gamma_\infty(K_p \times K_q) \leq p$. Using Propositions 5.5 and 6.7, it also holds that $\gamma_\infty(K_p \times K_q) \geq \beta(K_p \times K_q) = p$. The result therefore follows by a combination of these inequalities.

(b) Since $K_p \times K_q$ is $(p + q - 2)$–regular, the minimum vertex degree is $\delta = p + q - 2$. The result follows by way of Theorem 5.4 as $\gamma^*_\infty(K_p \times K_q) = n - \delta = pq - (p + q - 2)$. ■

In the case of the graph $P_p \times P_q$, the following result regarding the independence number is noted.

Proposition 6.9 *For any paths P_p and P_q, $p, q \geq 2$, $\beta(P_p \times P_q) = \lceil \frac{pq}{2} \rceil$.*

Proof: Denote the graph $P_p \times P_q$ by $\{v_{i,j} : \ i = 1, 2, \ldots, p, \ j = 1, 2, \ldots, q\}$. Since, clearly, $\beta(P_n) = \lceil \frac{n}{2} \rceil$ for any $n \in \mathbb{N}$, it follows that the set

$$\begin{aligned} I = &\left\{v_{2i-1,2j-1} : \ i = 1, 2, \ldots, \left\lceil \tfrac{p}{2} \right\rceil, j = 1, 2, \ldots, \left\lceil \tfrac{q}{2} \right\rceil\right\} \\ &\cup \left\{v_{2i,2j} : \ i = 1, 2, \ldots, \left\lfloor \tfrac{p}{2} \right\rfloor, j = 1, 2, \ldots, \left\lfloor \tfrac{q}{2} \right\rfloor\right\} \end{aligned}$$

is an independent set of $P_p \times P_q$, so that $\beta(P_p \times P_q) \geq \lceil \frac{p}{2} \rceil \lceil \frac{q}{2} \rceil + \lfloor \frac{p}{2} \rfloor \lfloor \frac{q}{2} \rfloor = \lceil \frac{pq}{2} \rceil$ (see Proposition A.5 in the appendix for a motivation of this equality). An example of such an independent set of $P_p \times P_q$, for $p = 5$ and $q = 9$, is illustrated in Figure 6.1.

Suppose $\beta(P_p \times P_q) > \lceil \frac{pq}{2} \rceil$. Then there exists a $j^* \in \{1, 2, \ldots, q - 1\}$ such that $\langle\{v_{i,j} : j = j^*, j^* + 1, \ i = 1, 2, \ldots, p\}\rangle$ contains at least $p + 1$ independent vertices, which is impossible. Hence $\beta(P_p \times P_q) \leq \lceil \frac{pq}{2} \rceil$ and the result follows. ■

As mentioned above, Burger *et al.* [3] provided the following result for the cartesian product $P_p \times P_q$.

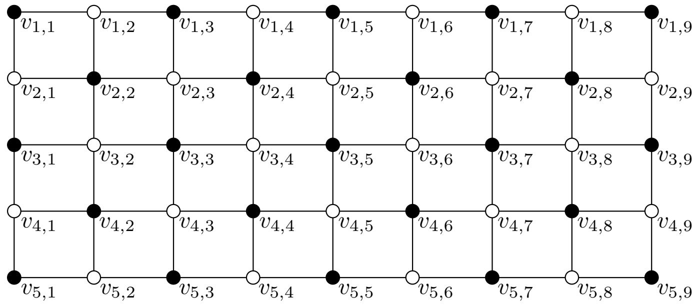


Figure 6.1: *The Cartesion product $P_5 \times P_9$, with an independent set displayed as dark vertices.*

Proposition 6.10 *For any paths P_p and P_q, $p, q \geq 2$,*

(a) $\gamma_\infty(P_p \times P_q) = \lceil \frac{pq}{2} \rceil$.

(b) $\gamma^*_\infty(P_p \times P_q) = pq - 2$.

Proof: (a) Denote the graph $P_p \times P_q$ by $\{v_{i,j} : \ i = 1, 2, \ldots, p, \ j = 1, 2, \ldots, q\}$ and consider two cases. Firstly, suppose p or q is even. Without loss of generality assume that p is even. Then

$$\mathcal{M}_1 = \bigcup_{j=1}^{q} \bigcup_{i=1}^{\frac{p}{2}} \langle v_{2i-1,j}, v_{2i,j} \rangle$$

is a perfect matching of $P_p \times P_q$, and hence $\gamma_\infty(P_p \times P_q) \leq \frac{pq}{2}$ by Corollary 5.2. Since $\beta(P_p \times P_q) = \frac{pq}{2}$ by Proposition 6.9, it follows from Proposition 5.5 that $\gamma_\infty(P_p \times P_q) \geq \frac{pq}{2}$. It therefore follows that

$$\gamma_\infty(P_p \times P_q) = \frac{pq}{2} \quad \text{if } p \text{ or } q \text{ is even.} \tag{6.12}$$

Suppose p and q are both odd. Then

$$\mathcal{M}_2 = \left(\bigcup_{j=1}^{q} \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor} \langle v_{2i-1,j}, v_{2i,j} \rangle \right) \cup \left(\bigcup_{j=1}^{q-2} \langle v_{p,j}, v_{p,j+1} \rangle \right)$$

is a perfect matching of the graph $P_p \times P_q - v_{p,q}$. By Corollary 5.2 it follows that $\gamma_\infty(P_p \times P_q) \leq \frac{pq+1}{2}$ and by Proposition 6.9 it follows that $\beta(P_p \times P_q) = \frac{pq+1}{2}$. Hence

$$\gamma_\infty(P_p \times P_q) = \frac{pq+1}{2} \quad \text{if } p \text{ and } q \text{ are odd.} \tag{6.13}$$

The desired result now follows by a combination of (6.12) and (6.13).

(b) The minimum degree for the graph $P_p \times P_q$ is $\delta = 2$, so that the desired result follows directly from Theorem 5.4. ■

Although Burger *et al.* [3] could only obtain exact values for the foolproof infinite order domination number for the cartesian product $C_p \times C_q$, bounds on the smart parameter were provided, as mentioned in Proposition 3.18(e) and (f). The following result regarding the independence number of $C_p \times C_q$ is noted.

Proposition 6.11 *For any cycles C_p and C_q, with $2 \le p \le q$,*

$$\beta(C_p \times C_q) = \min\left(p\left\lfloor\frac{q}{2}\right\rfloor, q\left\lfloor\frac{p}{2}\right\rfloor\right) = \begin{cases} p\left\lfloor\frac{q}{2}\right\rfloor & \text{if } p \text{ is even,} \\ q\left\lfloor\frac{p}{2}\right\rfloor & \text{if } p \text{ is odd.}\end{cases}$$

Proof: Denote the vertices of the graph $C_p \times C_q$ by $\{v_{i,j} : i = 1, 2, \ldots, p, j = 1, 2, \ldots, q\}$, such that

$$\langle v_{i,1}, v_{i,2}, \ldots, v_{i,q}\rangle \cong C_q \text{ for all } i = 1, 2, \ldots, p$$

and

$$\langle v_{1,j}, v_{2,j}, \ldots, v_{p,j}\rangle \cong C_p \text{ for all } j = 1, 2, \ldots, q.$$

Also note that

$$v_{i,j}v_{i+1,j+1} \notin E(C_p \times C_q) \text{ for any } i \in \{1, 2, \ldots, p-1\} \text{ and } j \in \{1, 2, \ldots, q-1\}$$

and

$$v_{i,j}v_{i-1,j+1} \notin E(C_p \times C_q) \text{ for any } i \in \{2, 3, \ldots, p\} \text{ and } j \in \{1, 2, \ldots, q-1\}.$$

Suppose p or q is even. It is clear that the set

$$\begin{aligned} I = &\left\{v_{2i-1,2j-1} : i = 1, 2, \ldots, \left\lfloor\tfrac{p}{2}\right\rfloor, j = 1, 2, \ldots, \left\lfloor\tfrac{q}{2}\right\rfloor\right\} \\ &\cup \left\{v_{2i,2j} : i = 1, 2, \ldots, \left\lfloor\tfrac{p}{2}\right\rfloor, j = 1, 2, \ldots, \left\lfloor\tfrac{q}{2}\right\rfloor\right\}\end{aligned}$$

is an independent set of $C_p \times C_q$. It follows that

$$\beta(C_p \times C_q) \ge 2\left\lfloor\frac{p}{2}\right\rfloor\left\lfloor\frac{q}{2}\right\rfloor \text{ if } p \text{ or } q \text{ is even.} \tag{6.14}$$

Suppose p and q are both odd, with $p \le q$. Let $m = \beta(C_p) = \lfloor\frac{p}{2}\rfloor$ and let the set $\{v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_m}\}$ denote a maximum independent set of C_p, with $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. Furthermore, let

$$I_1 = \left\{v_{\alpha_1+j-2 \pmod p)+1,j}, \ldots, v_{\alpha_m+j-2 \pmod p)+1,j} : j = 1, 2, \ldots, p\right\}$$

and

$$\begin{aligned} I_2 = &\{v_{\alpha_1,j}, v_{\alpha_2,j}, \ldots, v_{\alpha_m,j} : j = p+1, p+3, \ldots, q-1\} \\ &\cup \left\{v_{\alpha_1 \pmod p)+1,j}, \ldots, v_{\alpha_m \pmod p)+1,j} : j = p+2, p+4, \ldots, q\right\}.\end{aligned}$$

If $p = q$, then the set I_1 is an independent set of $C_p \times C_q$, whereas if $p < q$, then $I_1 \cup I_2$ is an independent set of $C_p \times C_q$. An example of such an independent set of $C_p \times C_q$, for $p = 5$ and $q = 9$, is illustrated in Figure 6.2.

It follows that

$$\beta(C_p \times C_q) \ge q\left\lfloor\frac{p}{2}\right\rfloor \text{ if } p \text{ and } q \text{ are odd.} \tag{6.15}$$

Since, clearly, $\beta(G) \le \beta(G-e)$ for any graph G and any edge $e \in E(G)$, and $\beta(C_n) = \lfloor\frac{n}{2}\rfloor$ for any $n \in \mathbb{N}$, it follows that

$$\beta(C_p \times C_q) \le \min\left(p\left\lfloor\frac{q}{2}\right\rfloor, q\left\lfloor\frac{p}{2}\right\rfloor\right) \tag{6.16}$$

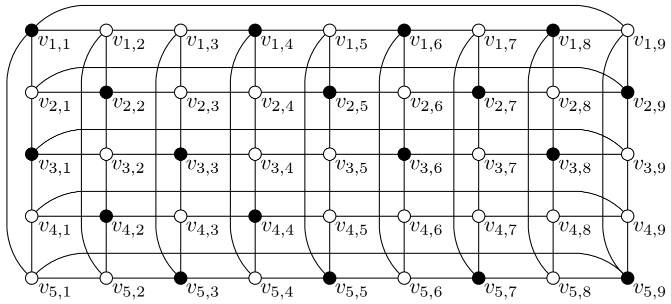


Figure 6.2: *The Cartesion product $C_5 \times C_9$, with an independent set displayed as dark vertices.*

for any $p, q \geq 2$, because $\cup_{j=1}^{q} C_p$ and $\cup_{i=1}^{p} C_q$ are both spanning subgraphs of $C_p \times C_q$. By a combination of (6.14), (6.15) and (6.16), the result follows. ■

In the case of the infinite order domination numbers, the lower bound stated in Proposition 3.18(e) is improved in the following proposition, but the problem of determining whether the upper bound is sharp if $p, q \geq 4$, remains open.

Proposition 6.12 *For any cycles C_p and C_q, with $p, q \geq 2$,*

(a) $\min(p\lfloor \frac{q}{2} \rfloor, q\lfloor \frac{p}{2} \rfloor) \leq \gamma_\infty(C_p \times C_q) \leq \lceil \frac{pq}{2} \rceil$,

(b) $\gamma_\infty^*(C_p \times C_q) = pq - 4$.

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Proof: (a) The lower bound follows from Proposition 6.11 and Proposition 5.5. Furthermore, by Corollary 4.2 and Proposition 6.10(a), it follows that $\gamma_\infty(C_p \times C_q) \leq \gamma_\infty(P_p \times P_q) = \lceil \frac{pq}{2} \rceil$, since $P_p \times P_q$ is a spanning subgraph of $C_p \times C_q$.

(b) The minimum degree for the graph $C_p \times C_q$ is $\delta = 4$. The result therefore follows directly from Theorem 5.4, with $n = pq$. ■

As in [3], it is noted that $\gamma_\infty(C_p \times C_q) = \frac{pq}{2}$ if p and q are even. Also, it is clear that $\beta(C_3 \times C_q) \geq q$ and $\mathfrak{c}(C_3 \times C_q) \leq q$ for any $q \in \mathbb{N}$, so that $\gamma_\infty(C_3 \times C_q) = q$, by equation (5.6) in §5.5, as mentioned by Goddard *et al.* [12]. Burger *et al.* [3] conjectured that the upper bound in Proposition 6.12 is sharp if $p, q \geq 4$. This problem remains open. It is, however, noted that the difference between the upper and lower bound grows as p and/or q increases. This is illustrated in Figure 6.3, with $p = 5$ and q increasing. The difference between the previously known bounds, that stated in Proposition 3.18(e), is included for comparison as the dotted line in Figure 6.3. Goddard *et al.* [12] proved, by means of a computer search, that $\gamma_\infty(C_p \times C_q) = \lceil \frac{pq}{2} \rceil$ for $q = 4$ and $p \in \{4, 5\}$. This does provide some evidence to support the conjecture.

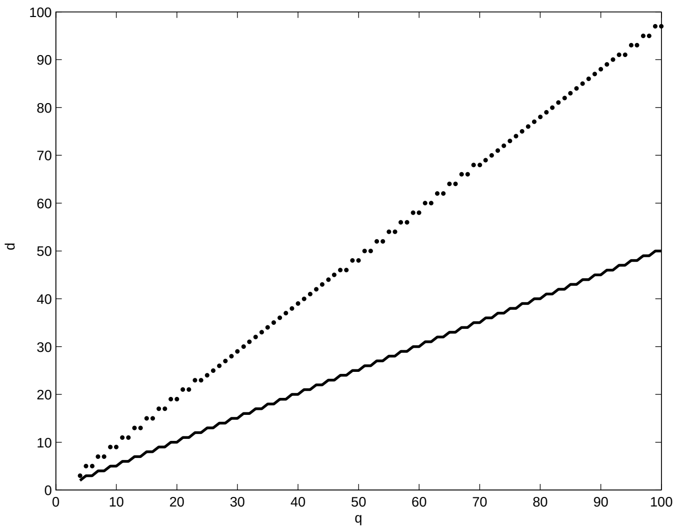

Figure 6.3: *An illustration of the difference $d = \lceil \frac{pq}{2} \rceil - \min(p\lfloor \frac{q}{2} \rfloor, q\lfloor \frac{p}{2} \rfloor)$ (indicated by the solid line) between the bounds in Proposition 6.12 with increasing q and $p = 5$. The difference between the previously known bounds are indicated by the dotted line.*

6.5 Complete Multipartite Graphs

As mentioned in §3.3 and §3.4, Cockayne *et al.* [8] established the values of the $(1,1)$– and $(2,1)$–smart domination numbers for the complete multipartite graph. These results, as well as the corresponding foolproof results were summarised by Benecke *et al.* [1]. By Corollary 4.1, the result holds for any number of guards allowed to be stationed at a vertex, since $\min(\Delta, k+1) \leq 2$ if $k = 1$, and therefore $\ell \in \{1,2\}$.

Proposition 6.13 *Consider the complete multipartite graph $K_{p_1,p_2,\ldots,p_t}$, $p_i \in \mathbb{N}$ $(i = 1,2,\ldots,t)$ and $p_i \leq p_{i+1}$ $(i = 1,2,\ldots,t-1)$.*

(a) If $\ell \in \{1,2\}$, $p_1 = 2$ and $t \geq 2$, or if $\ell = 2$, $p_1 = 1$ and $t \geq 2$, then

$$\gamma_{\ell,1}(K_{p_1,p_2,\ldots,p_t}) = \gamma^*_{\ell,1}(K_{p_1,p_2,\ldots,p_t}) = 2.$$

(b) If $p_1 = 1$, then

$$\gamma_{1,1}(K_{p_1,p_2}) = \gamma^*_{1,1}(K_{p_1,p_2}) = p_2.$$

(c) If $p_1 = 1$ and $t \geq 3$, then

$$\gamma_{1,1}(K_{p_1,p_2,\ldots,p_t}) = \gamma^*_{1,1}(K_{p_1,p_2,\ldots,p_t}) = \begin{cases} 2 & \text{if } p_2 \in \{1,2\} \\ 3 & \text{if } p_2 \geq 3. \end{cases}$$

(d) If $\ell \in \{1,2\}$ and $t, p_1 \geq 3$, then

$$\gamma_{\ell,1}(K_{p_1,p_2,\ldots,p_t}) = \gamma^*_{\ell,1}(K_{p_1,p_2,\ldots,p_t}) = 3.$$

(e) If $\ell \in \{1,2\}$, then

$$\gamma_{\ell,1}(K_{p_1,p_2}) = \gamma^*_{\ell,1}(K_{p_1,p_2}) = \begin{cases} 3 & \text{if } p_1 = 3 \\ 4 & \text{if } p_1 \geq 4. \end{cases} \qquad \blacksquare$$

With this result, the problem of finding the smart and foolproof finite order domination parameters, for the case $k = 1$, is completely resolved. The case where $k > 1$ is considered in the remainder of this section. Benecke *et al.* [1] resolved this case for when $\ell \in \{1, 2\}$, but it will be shown that these results may be generalised to hold for any $\ell \in \{1, 2, \ldots, \min(\Delta, k + 1)\}$. The graph under consideration will henceforth be denoted by $G \cong K_{p_1,p_2,\ldots,p_t}$, always assuming that $t \geq 2$ (unless explicitly stated otherwise) and $p_i \leq p_{i+1}$ $(i = 1, 2, \ldots, t-1)$. Furthermore, it will be assumed that $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$ and that the partite sets of G are denoted by $S_j = \{s_{j,1}, s_{j,2} \ldots, s_{j,p_j}\}$, so that $|S_j| = p_j$ for $j = 1, 2, \ldots, t$. The following two lemmas will prove useful.

Lemma 6.1 $\gamma^*_{\ell,k}(G) \leq p_t$.

Proof: Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G, with $V_1^{(0)} = S_t$ and $V_0^{(0)} = V(G)\backslash V_1^{(0)}$. For no sequence $v_0, v_1, \ldots, v_{k-1}$ does there exist a sequence $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$, with $u_i \in V_1^{(i)} \cap N[v_i]$, $i = 1, 2, \ldots, k-1$, such that $V_1^{(k)} \subset S_j$ for any $j \in \{1, 2, \ldots, t-1\}$, since $w(f^{(0)}) = p_t \geq p_j$ for all $j = 1, 2, \ldots, t$. Therefore $f^{(0)}$ is a $(1, k)$–FDF of G and it is concluded that $\gamma^*_{\ell,k}(G) \leq \gamma^*_{1,k}(G) \leq p_t$. $\blacksquare$

Lemma 6.2 $\gamma_{\ell,k}(G) \geq \min\{k + 1, p_t\}$.

Proof: Suppose to the contrary that $\gamma_{\ell,k}(G) = q$ for some $0 < q < \min\{k+1, p_t\}$. Then there exists an (ℓ, k)–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ of G with weight $w(f^{(0)}) = q$. Let $\mathcal{S} = \cup_{j=1}^{t-1} S_j$ and suppose $\sum_{j=1}^{\ell} j|\mathcal{S} \cap V_j^{(0)}| = c$ for some $c \leq q \leq k$. Consider a sequence of distinct, unoccupied vertices $v_0, v_1, \ldots, v_{c-1}$ in S_t. Such a sequence exists, since $q < p_t$. It follows that no sequence $u_i \in (V(G)\backslash V_0^{(i)}) \cap N(v_i)$, $i = 0, 1, \ldots, c-1$, can protect v_i, $i = 0, 1, \ldots, c-1$, under $f^{(0)}$, because $q < p_t$ implies that at least one vertex in S_t is left undominated under $f^{(c)}$. This contradiction yields the desired result. $\blacksquare$

The reader is also referred to Propositions 4.1 and 4.2, which will be used implicitly throughout this section. The proof of the main result of this section, summarised in Theorem 6.3, is divided into three subcases, depending on the size of k (informally seen as the number of attacks on the graph) relative to the partite set cardinalities of the complete multipartite graph in question.

6.5.1 Small Number of Attacks

In this section, the finite order domination numbers of a complete multipartite graph is considered when the size of k is small relative to the cardinality of the smallest partite set of the graph.

Proposition 6.14 *Suppose* $1 < k < p_1$.

(a) If $t(k+1) < (t-1)p_1$, *then* $\gamma_{\ell,k}(G) = \gamma^*_{\ell,k}(G) = \left\lceil \frac{t}{t-1}(k+1) \right\rceil$.

(b) If $t(k+1) \geq (t-1)p_1$, *then* $\gamma_{\ell,k}(G) = \gamma^*_{\ell,k}(G) = p_1$.

Proof: (a) Let $m = \lceil \frac{k+1}{t-1} \rceil (t-1) - (k+1)$. Take $V_1^{(0)}$ to be the set of any $\lfloor \frac{k+1}{t-1} \rfloor$ distinct vertices from each of any m partite sets of G, together with any $\lceil \frac{k+1}{t-1} \rceil$ distinct vertices from each of the remaining partite sets, for example

$$V_1^{(0)} = \left(\bigcup_{i=1}^{m} \left\{ s_{i,1}, s_{i,2}, \ldots, s_{i,\lfloor \frac{k+1}{t-1} \rfloor} \right\} \right) \bigcup \left(\bigcup_{i=m+1}^{t} \left\{ s_{i,1}, s_{i,2}, \ldots, s_{i,\lceil \frac{k+1}{t-1} \rceil} \right\} \right),$$

and consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G, with $V_0^{(0)} = V(G) \backslash V_1^{(0)}$. Note that for any collection of $t-1$ partite sets of G,

$$\left| \left(\left(\bigcup_{j=1}^{t} S_j \right) \backslash S_d \right) \cap V_1^{(0)} \right| \geq m \left\lfloor \frac{k+1}{t-1} \right\rfloor + (t-1-m) \left\lceil \frac{k+1}{t-1} \right\rceil = k+1$$

for any $d \in \{1, 2, \ldots, t\}$, by using Proposition A.6. It is therefore impossible to attain a guard function $f^{(k)} = (V_0^{(k)}, V_1^{(k)})$ from $f^{(0)}$, such that $V_1^{(k)} \subseteq S_d$ for some $d \in \{1, 2, \ldots, t\}$. Hence $f^{(0)}$ is a $(1, k)$–FDF of G with weight

$$\begin{aligned} w\left(f^{(0)}\right) &= \left|V_1^{(0)}\right| = m \left\lfloor \frac{k+1}{t-1} \right\rfloor + (t-m) \left\lceil \frac{k+1}{t-1} \right\rceil \\ &= k+1+ \left\lceil \frac{k+1}{t-1} \right\rceil \\ &= \left\lceil \frac{t}{t-1}(k+1) \right\rceil, \end{aligned}$$

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and it is concluded that

$$\gamma_{\ell,k}(G) \leq \gamma^*_{\ell,k}(G) \leq \gamma^*_{1,k}(G) \leq \left\lceil \frac{t}{t-1}(k+1) \right\rceil. \tag{6.17}$$

Suppose $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ, k)–SDF of G with weight $w(f^{(0)}) < \lceil \frac{t}{t-1}(k+1) \rceil$. Let $\mathcal{S}_d = (\cup_{j=1}^{t} S_j) \backslash S_d$ for some $d \in \{1, 2, \ldots, t\}$, and note that, if $\sum_{j=1}^{\ell} j |\mathcal{S}_d \cap V_j^{(0)}| \geq k+1$ for every $d \in \{1, 2, \ldots, t\}$ in the deployment of $f^{(0)}$, then $\sum_{j=1}^{\ell} j |V_j^{(0)}| \geq \lceil \frac{t}{t-1}(k+1) \rceil > w(f^{(0)})$ in G. This contradiction shows that there exist $t-1$ partite sets of G, such that $\sum_{j=1}^{\ell} j |\mathcal{S}_d \cap V_j^{(0)}| \leq k$, for some $d \in \{1, 2, \ldots, t\}$, under $f^{(0)}$. Without loss of generality, let these partite sets be $S_2, S_3, \ldots, S_t$, i.e. let $d = 1$, and suppose $\sum_{j=1}^{\ell} j |\mathcal{S}_d \cap V_j^{(0)}| = c$ for some $c \leq k$. Consider a sequence of distinct, unoccupied vertices $v_0, v_1, \ldots, v_{c-1}$ in S_1. Such a sequence exists, because the inequality $t(k+1) < (t-1)p_1$ implies that

$$\left\lceil \frac{t}{t-1}(k+1) \right\rceil - 1 < \frac{t(k+1)}{t-1} < p_1.$$

It follows that no sequence $u_i \in (V(G)\backslash V_0^{(i)}) \cap N(v_i)$, $i = 0, 1, \ldots, c-1$, can protect v_i, $i = 0, 1, \ldots, c-1$, under $f^{(0)}$, since at least one vertex in S_1 is not dominated under $f^{(c)}$. This contradiction shows that

$$\gamma^*_{\ell,k}(G) \geq \gamma_{\ell,k}(G) \geq \left\lceil \frac{t}{t-1}(k+1) \right\rceil. \tag{6.18}$$

The result therefore follows by a combination of (6.17) and (6.18).

(b) Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G, with $V_1^{(0)} = S_1$ and $V_0^{(0)} = V(G)\backslash V_1^{(0)}$. Since $p_1 > k$, it is impossible to attain a guard function $f^{(k)} = (V_0^{(k)}, V_1^{(k)})$, such that $V_1^{(k)} \cap S_1 = \emptyset$. Therefore it follows that $f^{(0)}$ is a $(1,k)$–FDF of G, and hence

$$\gamma_{\ell,k}(G) \leq \gamma^*_{\ell,k}(G) \leq \gamma^*_{1,k}(G) \leq p_1. \tag{6.19}$$

Now suppose $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ,k)–SDF of G with weight $w(f^{(0)}) < p_1$. Let $\mathcal{S}_d = (\cup_{j=1}^t S_j)\backslash S_d$ for some $d \in \{1, 2, \ldots, t\}$, and note that, if $\sum_{j=1}^\ell j|\mathcal{S}_d \cap V_j^{(0)}| \geq k+1$ for every $d \in \{1, 2, \ldots, t\}$ in the deployment of $f^{(0)}$, then $\sum_{j=1}^\ell j|V_j^{(0)}| \geq \lceil \frac{t}{t-1}(k+1) \rceil \geq \frac{t}{t-1}(k+1) \geq p_1 > w(f^{(0)})$ in G. This contradiction shows that there exist $t-1$ partite sets of G, such that $\sum_{j=1}^\ell j|\mathcal{S}_d \cap V_j^{(0)}| \leq k$, for some $d \in \{1, 2, \ldots, t\}$, under $f^{(0)}$. Without loss of generality, let these partite sets be $S_2, S_3, \ldots, S_t$, i.e. let $d = 1$, and suppose $\sum_{j=1}^\ell j|\mathcal{S}_d \cap V_j^{(0)}| = c$ for some $c \leq k$. Consider a sequence of distinct, unoccupied vertices $v_0, v_1, \ldots, v_{c-1}$ in S_1. Such a sequence exists, since $w(f^{(0)}) < p_1$. It follows that no sequence $u_i \in (V(G)\backslash V_0^{(i)}) \cap N(v_i)$, $i = 0, 1, \ldots, c-1$, can protect v_i, $i = 0, 1, \ldots, c-1$, under $f^{(0)}$, since at least one vertex in S_1 is not dominated under $f^{(c)}$. This contradiction shows that

$$\gamma^*_{\ell,k}(G) \geq \gamma_{\ell,k}(G) \geq p_1. \tag{6.20}$$

The result therefore follows by a combination of (6.19) and (6.20). ■

6.5.2 Intermediate Number of Attacks

Values for the finite order domination numbers of a complete multipartite graph for an intermediate number of attacks relative to the partite set cardinalities of the graph are explored next. The number of attacks, k, is considered intermediate if $p_1 \leq k < p_{t-1}$. Assume $p_{i^*} \leq k < p_{i^*+1}$ for some $i^* \in \{1, 2, \ldots, t-1\}$. The first result in this section caters for the case where the first i^* partite sets are sufficiently large, while the second result in this section regards the opposite situation.

Proposition 6.15 *If there exists an $i^* \in \{1, 2, \ldots, t-1\}$ such that $p_{i^*} \leq k < p_{i^*+1}$, and if $\ell \sum_{j=1}^{i^*} p_j \geq k+1$, then*

$$\gamma_{\ell,k}(G) = \gamma^*_{\ell,k}(G) = k+1.$$

Proof: By considering two cases, it is first shown that $\gamma^*_{\ell,k}(G) \leq k+1$.

Suppose $\sum_{j=1}^{i^*} p_j < k+1$. Define $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ as a guard function of G with weight $w(f^{(0)}) = k+1$, such that $\cup_{j=1}^{\ell} V_j^{(0)} = \cup_{j=1}^{i^*} S_j$. Such a safe guard function $f^{(0)}$ can be constructed, since $\sum_{j=1}^{i^*} p_j < k+1 \leq \ell \sum_{j=1}^{i^*} p_j$. Note that $\cup_{j=1}^{i^*} S_j \subseteq V(G)\backslash V_0^{(0)}$, and it is impossible to attain a guard function $f^{(k)} = (V_0^{(k)}, V_1^{(k)}, \ldots, V_\ell^{(k)})$, such that $V(G)\backslash V_0^{(k)} \subseteq S_d$ for some $d \in \{i^*+1, \ldots, t\}$, since $w(f^{(0)}) = k+1$. It is therefore concluded that $f^{(0)}$ is an (ℓ, k)–FDF of G.

Now suppose $\sum_{j=1}^{i^*} p_j \geq k+1$. Let m be such that $\sum_{j=1}^{m} p_j < k+1 \leq \sum_{j=1}^{m+1} p_j$. Take $V_1^{(0)}$ as the set of all vertices from the smallest m partite sets of G, as well as $k+1-\sum_{j=1}^{m} p_j$ distinct vertices from S_{m+1}, and define $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$, with $V_0^{(0)} = V(G)\backslash V_1^{(0)}$ and $w(f^{(0)}) = k+1$. Note that $\cup_{j=1}^{m} S_j \subset V_1^{(0)}$ under $f^{(0)}$, and it is impossible to attain a guard function $f^{(k)} = (V_0^{(k)}, V_1^{(k)})$, such that $V_1^{(k)} \subseteq S_d$ for some $d \in \{m+1, \ldots, t\}$, since $p_{m+1} \leq p_{i^*} < k+1 = w(f^{(0)})$. Hence $f^{(0)}$ is a $(1,k)$–FDF, and therefore also an (ℓ, k)–FDF of G.

In both cases it holds that

$$\gamma_{\ell,k}(G) \leq \gamma^*_{\ell,k}(G) \leq k+1. \tag{6.21}$$

Furthermore, since $k < p_{i^*+1} \leq p_t$, it follows from Lemma 6.2 that

$$\gamma^*_{\ell,k}(G) \geq \gamma_{\ell,k}(G) \geq k+1. \tag{6.22}$$

The result therefore follows by a combination of (6.21) and (6.22). ■

Informally stated, if $\ell \sum_{j=1}^{i^*} p_j \geq k+1$, then it is possible to deploy $k+1$ guards onto vertices in the smallest i^* partite sets. Such a deployment, discussed in the proof of Proposition 6.15, protects the graph against k attacks, since it is impossible to move all $k+1$ guards to one of the larger partite sets. The opposite scenario, where $\ell \sum_{j=1}^{i^*} p_j < k+1$, is considered in the following proposition, and hence all possibilities are catered for in the case of protection against an intermediate number of attacks.

Proposition 6.16 *Suppose there exists an $i^* \in \{1, 2, \ldots, t-2\}$ such that $p_{i^*} \leq k < p_{i^*+1}$ and let $t \geq 3$. If $\ell \sum_{j=1}^{i^*} p_j < k+1$, then*

$$\gamma_{\ell,k}(G) = \gamma^*_{\ell,k}(G) = \min\{p_{i^*+1}, \sigma_{i^*}\},$$

where

$$\sigma_{i^*} = \left\lceil \frac{t-i^*}{t-i^*-1}(k+1) - \frac{\ell}{t-i^*-1} \sum_{j=1}^{i^*} p_j \right\rceil.$$

Proof: Two cases are considered. Suppose first that $\sigma_{i^*} < p_{i^*+1}$, and let

$$m = \left\lceil \frac{k+1-\ell \sum_{j=1}^{i^*} p_j}{t-i^*-1} \right\rceil (t-i^*-1) - \left(k+1-\ell \sum_{j=1}^{i^*} p_j\right).$$

Let $W_1 = \cup_{j=1}^{i^*} S_j$, the set of all vertices in the i^* smallest partite sets of G and let W_2 be a set of $\lfloor \frac{k+1-\ell\sum_{i=1}^{i^*} p_i}{t-i^*-1} \rfloor$ distinct vertices from each of the m partite sets $S_{i^*+1}, \ldots, S_{i^*+m}$, together with $\lceil \frac{k+1-\ell\sum_{i=1}^{i^*} p_i}{t-i^*-1} \rceil$ distinct vertices from the $(t - i^* - m)$ largest partite sets of G, that is

$$W_2 = \left(\bigcup_{j=i^*+1}^{i^*+m} \left\{ s_{j,1}, s_{j,2}, \ldots, s_{j,\left\lfloor \frac{k+1-\ell\sum_{i=1}^{i^*} p_i}{t-i^*-1} \right\rfloor} \right\} \right) \bigcup \left(\bigcup_{j=i^*+m+1}^{t} \left\{ s_{j,1}, s_{j,2}, \ldots, s_{j,\left\lceil \frac{k+1-\ell\sum_{i=1}^{i^*} p_i}{t-i^*-1} \right\rceil} \right\} \right).$$

Such a set W_2 exists, since $\lceil \frac{k+1-\ell\sum_{i=1}^{i^*} p_i}{t-i^*-1} \rceil \leq k < p_{i^*+1} \leq p_j$ for all $j \in \{i^*+1, \ldots, t\}$. Define the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ of G, with $V_\ell^{(0)} = W_1$ if $\ell > 1$,

$$V_1^{(0)} = \begin{cases} W_1 \cup W_2 & \text{if } \ell = 1 \\ W_2 & \text{otherwise,} \end{cases}$$

and $V_0^{(0)} = V(G) \backslash (V_1^{(0)} \cup V_\ell^{(0)})$. Let $\mathcal{S}_d = (\cup_{j=1}^{t} S_j) \backslash S_d$, $d \in \{i^*+1, \ldots, t\}$, and note that, for any collection of $t-1$ partite sets of G that includes the smallest i^* sets, the cardinality of $\mathcal{S}_d \cap (V(G) \backslash V_0^{(0)})$ is at least

$$m \left\lfloor \frac{k+1-\ell\sum_{j=1}^{i^*} p_j}{t-i^*-1} \right\rfloor + (t-i^*-1-m) \left\lceil \frac{k+1-\ell\sum_{j=1}^{i^*} p_j}{t-i^*-1} \right\rceil + \ell \sum_{j=1}^{i^*} p_j = k+1$$

for every $d \in \{i^*+1, \ldots, t\}$, by using Proposition A.6. It is therefore impossible to obtain a guard function $f^{(k)} = (V_0^{(k)}, V_1^{(k)}, \ldots, V_\ell^{(k)})$ from $f^{(0)}$, such that $V(G) \backslash V_0^{(k)} \subseteq S_d$, for some $d \in \{i^*+1, \ldots, t\}$. Hence $f^{(0)}$ is an (ℓ, k)–FDF of G of weight

$$\begin{aligned} w\left(f^{(0)}\right) &= \ell \left| V_\ell^{(0)} \cap W_1 \right| + \left| V_1^{(0)} \cap W_2 \right| \\ &= \ell \sum_{j=1}^{i^*} p_j + m \left\lfloor \frac{k+1-\ell\sum_{j=1}^{i^*} p_j}{t-i^*-1} \right\rfloor + (t-i^*-m) \left\lceil \frac{k+1-\ell\sum_{j=1}^{i^*} p_j}{t-i^*-1} \right\rceil \\ &= \ell \sum_{j=1}^{i^*} p_j + k + 1 - \ell \sum_{j=1}^{i^*} p_j + \left\lceil \frac{k+1-\ell\sum_{j=1}^{i^*} p_j}{t-i^*-1} \right\rceil \\ &= k + 1 + \left\lceil \frac{k+1-\ell\sum_{j=1}^{i^*} p_j}{t-i^*-1} \right\rceil \\ &= \sigma_{i^*}, \end{aligned}$$

and it is concluded that

$$\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}^*(G) \leq \sigma_{i^*}. \tag{6.23}$$

Suppose $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ, k)–SDF of G with weight $w(f^{(0)}) < \sigma_{i^*}$. Let $\mathcal{S}_d = (\cup_{j=1}^{t} S_j) \backslash S_d$, $d \in \{i^*+1, \ldots, t\}$, and note that, if $\sum_{j=1}^{\ell} j |\mathcal{S}_d \cap V_j^{(0)}| \geq k+1$

in the deployment of $f^{(0)}$, for every $d \in \{i^*+1,\dots,t\}$, then $\sum_{j=1}^{\ell} j|V_j^{(0)}| \geq \sigma_{i^*} > w(f^{(0)})$ in G. This contradiction shows that there exist $t-1$ partite sets $\mathcal{S}_d$ of G, for some $d \in \{i^*+1,\dots,t\}$, such that $\sum_{j=1}^{\ell} j|\mathcal{S}_d \cap V_j^{(0)}| \leq k$ under $f^{(0)}$. Suppose $\sum_{j=1}^{\ell} j|\mathcal{S}_d \cap V_j^{(0)}| = c$ for some $c \leq k$ and consider a sequence of distinct, unoccupied problem vertices $v_0, v_1, \dots, v_{c-1}$ in S_d. Such a sequence exists due to the requirement that $w(f^{(0)}) < p_{i^*+1} \leq p_d$. It follows that no sequence $u_i \in (V(G)\backslash V_0^{(i)}) \cap N(v_i)$, $i = 0,1,\dots,c-1$, can protect v_i, $i = 0,1,\dots,c-1$, under $f^{(0)}$, because $\sigma_{i^*} - 1 < p_{i^*+1} \leq p_d$ for every $d \in \{i^*+1, i^*+2,\dots,t\}$, from which it follows that at least one vertex in S_d is left undominated under $f^{(c)}$. This contradiction shows that

$$\gamma^*_{\ell,k}(G) \geq \gamma_{\ell,k}(G) \geq \sigma_{i^*}. \tag{6.24}$$

The result therefore follows by a combination of (6.23) and (6.24).

Now suppose $\sigma_{i^*} \geq p_{i^*+1}$ and consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of G, with $V_1^{(0)} = S_{i^*+1}$ and $V_0^{(0)} = V(G)\backslash V_1^{(0)}$. Since $p_{i^*+1} > k$, it is impossible to obtain a safe guard function $f^{(k)} = (V_0^{(k)}, V_1^{(k)})$, such that $V_1^{(k)} \cap S_{i^*+1} = \emptyset$, and it is concluded that

$$\gamma_{\ell,k}(G) \leq \gamma^*_{\ell,k}(G) \leq \gamma^*_{1,k}(G) \leq p_{i^*+1}. \tag{6.25}$$

Suppose $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \dots, V_\ell^{(0)})$ is an (ℓ,k)–SDF of G with weight $w(f^{(0)}) < p_{i^*+1}$. By the same argument as used in the previous case, it follows that a sequence v_i, $i = 0,1,\dots,c-1$, of distinct vertices exists for some $1 \leq c \leq k$, the protection of which cannot result in a safe guard function $f^{(c)}$ of G, because $p_{i^*+1} - 1 < p_d$ for any $d \in \{i^*+1, i^*+2,\dots,t\}$. This contradiction shows that

$$\gamma^*_{\ell,k}(G) \geq \gamma_{\ell,k}(G) \geq p_{i^*+1}. \tag{6.26}$$

The result therefore follows by a combination of (6.25) and (6.26). ■

6.5.3 Large Number of Attacks

Finally, consider values for the finite order domination numbers of a complete multipartite graph for a large number of attacks relative to the partite set cardinalities of the graph. The number of attacks, k, is considered large if $k \geq p_{t-1}$. To avoid duplicity in the proofs of results, the case where $p_{t-1} \leq k < p_t$ and $\ell \sum_{j=1}^{t-1} p_j \geq k+1$ is included in Proposition 6.15. The remaining possibilities are catered for in the following proposition.

Proposition 6.17 *If $p_{t-1} \leq k < p_t$ and $\ell \sum_{j=1}^{t-1} p_j < k+1$, or if $k \geq p_t$, then $\gamma_{\ell,k}(G) = \gamma^*_{\ell,k}(G) = p_t$.*

Proof: By Lemma 6.1 it is known that

$$\gamma_{\ell,k}(G) \leq \gamma^*_{\ell,k}(G) \leq p_t. \tag{6.27}$$

Suppose $p_{t-1} \leq k < p_t$ and $\ell \sum_{j=1}^{t-1} p_j < k+1$, and that $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is any (ℓ, k)–SDF of G with weight $w(f^{(0)}) < p_t$. Note that, for the partite sets $S_1, S_2, \ldots, S_{t-1}$, it holds that $\sum_{j=1}^{\ell} j|(\cup_{d=1}^{t-1} S_d) \cap V_j^{(0)}| \leq k$, since the largest possible value is $\ell \sum_{j=1}^{t-1} p_j < k+1$. Suppose $\sum_{j=1}^{\ell} j|(\cup_{d=1}^{t-1} S_d) \cup V_j^{(0)}| = c$ for some $c \leq k$ and consider a sequence of distinct, unoccupied vertices $v_0, v_1, \ldots, v_{c-1}$ in S_t. Such a sequence exists, since $w(f^{(0)}) < p_t$. It follows that no sequence $u_i \in (V(G) \backslash V_0^{(i)}) \cap N(v_i)$, $i = 0, 1, \ldots, c-1$, can protect v_i, $i = 0, 1, \ldots, c-1$, under $f^{(0)}$, because $w(f^{(0)}) < p_t$ implies that at least one vertex in S_t is left undominated under $f^{(c)}$. This contradiction shows that $\gamma_{\ell,k}(G) \geq p_t$.

Suppose $k \geq p_t$. Then it follows from Lemma 6.2 that $\gamma_{\ell,k}(G) \geq p_t$. In both cases we have

$$\gamma_{\ell,k}^*(G) \geq \gamma_{\ell,k}(G) \geq p_t, \tag{6.28}$$

so that the result follows by a combination of (6.27) and (6.28). ■

6.5.4 Main Result

The results of §6.5.1–6.5.3 may be summarised in the following theorem, which is the main result of this section.

Pectora roborant cultus recti

Theorem 6.3 *For the complete multipartite graph $K_{p_1,p_2,\ldots,p_t}$, $t \geq 2$ and $k \geq 2$,*

$$\begin{aligned}&\gamma_{\ell,k}(K_{p_1,p_2,\ldots,p_t})\\ &\quad = \gamma_{\ell,k}^*(K_{p_1,p_2,\ldots,p_t})\end{aligned} = \begin{cases} \left\lceil \dfrac{t}{t-1}(k+1) \right\rceil & \text{if } 1 < k \leq \left\lceil \dfrac{t-1}{t} p_1 - 2 \right\rceil \\ p_1 & \text{if } \left\lceil \dfrac{t-1}{t} p_1 - 2 \right\rceil + 1 \leq k < p_1 \\ k+1 & \text{if } p_i \leq k < p_{i+1} \text{ and } \ell \displaystyle\sum_{j=1}^{i} p_j \geq k+1, \\ & \quad i \in \{1, 2, \ldots, t-1\} \\ \min\{p_{i+1}, \sigma_i\} & \text{if } p_i \leq k < p_{i+1} \text{ and } \ell \displaystyle\sum_{j=1}^{i} p_j < k+1, \\ & \quad i \in \{1, 2, \ldots, t-2\}, t \geq 3 \\ p_t & \text{if } p_{t-1} \leq k < p_t \text{ and } \ell \displaystyle\sum_{j=1}^{t-1} p_j < k+1, \\ & \quad \text{or if } k \geq p_t, \end{cases}$$

where $k \in \mathbb{N}$, $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$, $p_j \leq p_{j+1}$ $(j = 1, 2, \ldots, t-1)$ and

$$\sigma_i = \left\lceil \frac{t-i}{t-i-1}(k+1) - \frac{\ell}{t-i-1} \sum_{j=1}^{i} p_j \right\rceil. \qquad ■$$

Both smart and foolproof infinite order domination numbers of the complete multipartite graph were found by Burger *et al.* [3]. These trivially follow from the last case in Theorem 6.3.

Corollary 6.4 *For the complete multipartite graph $K_{p_1,p_2,\ldots,p_t}$, with $p_1 \leq p_2 \leq \cdots \leq p_t$, $\gamma_\infty(K_{p_1,p_2,\ldots,p_t}) = \gamma^*_\infty(K_{p_1,p_2,\ldots,p_t}) = p_t$, for all $t \geq 2$.*

Proof: By Proposition 4.1 and Theorem 5.4 it follows that

$$\gamma_\infty(K_{p_1,p_2,\ldots,p_t}) \leq \gamma^*_\infty(K_{p_1,p_2,\ldots,p_t}) = n - \delta = \sum_{i=1}^{t} p_i - \sum_{i=1}^{t-1} p_i = p_t. \tag{6.29}$$

However, from Propositions 4.1 and 4.3(a), Definition 5.1 and Theorem 6.3, it follows that

$$\gamma^*_\infty(K_{p_1,p_2,\ldots,p_t}) \geq \gamma_\infty(K_{p_1,p_2,\ldots,p_t}) \geq \gamma_{\ell,k}(K_{p_1,p_2,\ldots,p_t}) = p_t, \tag{6.30}$$

since $k \geq p_t$. The desired result is obtained by a combination of (6.29) and (6.30). ■

6.6 Trees

This section may be described as an introductory exploration of the smart finite order domination numbers for the special graph class of trees. The results contained in §6.6.1 and §6.6.2 were obtained by Henning [18] and are only slightly modified to cohere with the general notation of this thesis. In §6.6.3, the smart higher order domination numbers are examined for some special classes of trees, namely caterpillars and spiders.

6.6.1 A Lower Bound on $\gamma_{\ell,k}$

In §4.4, necessary conditions and sufficient conditions were given for when $\gamma_{\ell,k}(G) = \gamma(G)$, but when G is a tree, a characterisation for this equality may be obtained. Although the proof strategy provided by Henning [18] remains the same, this result may be generalised to hold true for any number of guards allowed per vertex.

A family $\mathcal{T}$ of trees will be constructed as follows: Suppose $a \in \mathbb{N}$ and let v_i be the centre of the star $T_i = K_{1,n_i}$, for some $n_i \geq 2$ and all $i = 1, 2, \ldots, a$. Let $S_A = \{v_1, v_2, \ldots, v_a\}$ and let L_A denote the set of all leaves of these a stars. Let $b \in \mathbb{N}$, such that $b \geq (\sum_{i=1}^{a} n_i) - a + 1$, let $T_0 = \cup_{j=1}^{b} K_2$ and let S_B be an independent set of b vertices in T_0 (i.e. one vertex from each copy of K_2 in T_0). Finally, let T be a tree obtained from $\cup_{i=0}^{a} T_i$ by adding $a + b - 1$ edges in such a way that

(i) each added edge joins vertices in $L_A \cup S_B$,

(ii) each vertex in L_A is adjacent to at least one vertex in S_B,

(iii) each vertex in S_B is incident with at least one added edge.

An example of such a tree is shown in Figure 6.4, with $a = 2$, $n_1 = 3$, $n_2 = 4$ and $b = 8$. Note that $|S_B| \geq \sum_{i=1}^{a} n_i - a + 1$ in order for (ii) and (iii) to hold, and that exactly $a + b - 1$ edges have to be added in order for the resulting graph to be both connected and acyclic. Let $\mathcal{T}$ denote the family of all such trees T. This notation will be used in the following propositions.

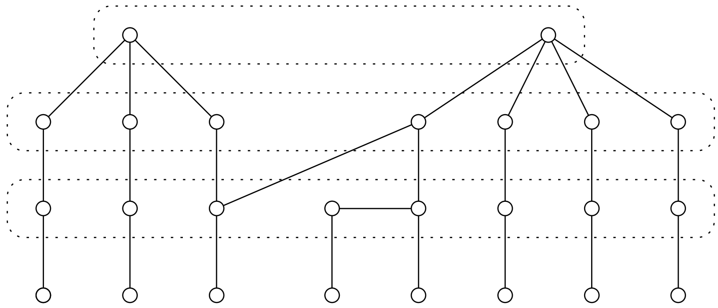


Figure 6.4: *An example of a tree belonging to the family $\mathcal{T}$, with $a = 2$, $n_1 = 3$, $n_2 = 4$ and $b = 8$.*

Proposition 6.18 *If $T \in \mathcal{T}$, then $\gamma_{\ell,1}(T) = \gamma(T)$ for any $\ell \in \{1, \ldots, \min(\Delta, 2)\}$.*

Proof: Suppose $T \in \mathcal{T}$. From the construction of the family $\mathcal{T}$ it is clear that $S = S_A \cup S_B$ is a minimum dominating set of T and that S_A is a packing in T. For every $v \in S_B$, the set $\text{epn}(v, S)$ consists only of the leaf adjacent to v in T. Furthermore, $\text{epn}(u, S) = \emptyset$ for every $u \in S_A$, so that for every $w \in L_A$, there exists a $u \in S_A$ such that $\langle \text{epn}(u, S) \cup \{u, w\} \rangle \cong K_2$. It follows from Theorem 4.2 that $\gamma_{\ell,1}(T) = \gamma(T)$. ■

Proposition 6.19 *If T is the corona of a tree and $k \in \mathbb{N}$, then $\gamma_{\ell,k}(T) = \gamma(T)$ for any $\ell \in \{1, 2, \ldots, \min(\Delta, k + 1)\}$.*

Proof: Suppose T is the corona of a tree, and let S be the set of all leaves of T. Clearly S is a minimum dominating set of T. Since every $u \in S$ is a leaf of T and T is the corona of a tree, it follows that $\langle \text{epn}(u, S) \cup \{u\} \rangle$ is isomorphic to K_2 and that every vertex $v \in V(T) \backslash S$ is a private neighbour of a vertex $u \in S$. By utilising Proposition 4.10, it is concluded that $\gamma_{\ell,k}(T) = \gamma(T)$ for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k + 1)\}$. ■

Propositions 6.18 and 6.19 may now be used to prove the following result, which provides the above mentioned characterisation. In the proof of this result, the sets S_A, L_A and S_B are defined in terms of the tree T and appropriate dominating function, and it is shown that T belongs to the family $\mathcal{T}$, in the case where $k = 1$.

Theorem 6.4 *A tree T satisfies $\gamma_{\ell,k}(T) = \gamma(T)$ for any $\ell \in \{1, 2, \ldots, \min(\Delta, k + 1)\}$ if and only if $k = 1$ and $T \in \mathcal{T}$, or $k \in \mathbb{N}$ and T is the corona of a tree.*

Proof: Suppose that $\gamma_{\ell,k}(T) = \gamma(T)$. Then, from Proposition 4.8, it holds that $\gamma_{1,k}(T) = \gamma(T)$. Consider a minimum weight $(1, k)$–SDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ of T. It may be assumed that $V_1^{(0)}$ contains no leaf, since otherwise every leaf in $V_1^{(0)}$ may simply be replaced with its neighbour. Thus it may be assumed that every vertex in $V_1^{(0)}$ has degree at least 2 in T. Let $S_B = \{u \in V_1^{(0)} : \operatorname{epn}(u, V_1^{(0)}) \neq \emptyset\}$ and $S_A = V_1^{(0)} \backslash S_B$. By Proposition 4.9, $\operatorname{epn}(u, V_1^{(0)})$ induces a clique for every $u \in V_1^{(0)}$. Since T is a tree and therefore acyclic, $|\operatorname{epn}(u, V_1^{(0)})| = 1$ for every $u \in S_B$. Let $S_B' = \{\operatorname{epn}(u, V_1^{(0)}) : u \in S_B\}$. Then $|S_B'| = |S_B|$. The rest of the proof is divided into six observations.

Observation 1 *S_A is a packing in T.*

Proof: Since $\operatorname{epn}(u, V_1^{(0)}) = \emptyset$ for every $u \in S_A$, it holds that no vertex u in S_A is adjacent to any other vertex in $V_1^{(0)}$, since otherwise $V_1^{(0)} \backslash \{u\}$ would be a dominating set of T with cardinality less than $\gamma(T)$. It therefore follows that $N(S_A) \subseteq V(T) \backslash V_1^{(0)}$. Furthermore, there does not exist a $v \in N(S_A)$ which is adjacent to two vertices $u_1, u_2 \in S_A$, since otherwise the set $(V_1^{(0)} \backslash \{u_1, u_2\}) \cup \{v\}$ would be a dominating set of T with cardinality less than $\gamma(T)$. Hence S_A is a packing in T. □

Observation 2 $V(T) \backslash (V_1^{(0)} \cup S_B') = N(S_A)$.

Proof: Since S_A is a packing in T and $\operatorname{epn}(u, V_1^{(0)}) = \emptyset$ for every $u \in S_A$, it follows that $N(S_A) \subseteq V(T) \backslash (V_1^{(0)} \cup S_B')$. Let $v \in V(T) \backslash (V_1^{(0)} \cup S_B')$. Then there exists a $u \in N(v) \cap V_1^{(0)}$ such that $f^{(1)} = \operatorname{move}(f^{(0)}, u \to v)$ is a safe guard function of T. It holds that $u \notin S_B$, since otherwise a vertex in S_B' would be left undominated under $f^{(1)}$. Therefore $u \in S_A$ and $v \in N(S_A)$, so that $V(T) \backslash (V_1^{(0)} \cup S_B') \subseteq N(S_A)$. □

Observation 3 *Each vertex in $N(S_A)$ is adjacent to only one vertex in S_A and to at least one vertex in S_B.*

Proof: By Observation 1, S_A is a packing in T, and so each vertex in $N(S_A)$ is necessarily adjacent to only one other vertex in S_A. Since $\operatorname{epn}(u, V_1^{(0)}) = \emptyset$ for every $u \in S_A$, it follows that each vertex in $N(S_A)$ is adjacent to at least one vertex in S_B. □

Observation 4 *S_B' is an independent set.*

Proof: Let $u_1, u_2 \in S_B$ and $\operatorname{epn}(u_i, V_1^{(0)}) = \{v_i\}$, $i = 1, 2$. Suppose v_1 is adjacent to v_2 in T. Since $\deg_T u_1 \geq 2$, there exists a vertex $w \in N(u_1) \backslash \{v_1\}$. If $w \in V_1^{(0)}$, then $(V_1^{(0)} \backslash \{u_1, u_2\}) \cup \{v_2\}$ is a dominating set of T with cardinality less than $\gamma(T)$. Hence $w \in V(T) \backslash (V_1^{(0)} \cup S_B') = N(S_A)$. Let u be the vertex in S_A adjacent to w. Since $\operatorname{epn}(u, V_1^{(0)}) = \emptyset$, the set $(V_1^{(0)} \backslash \{u, u_1, u_2\}) \cup \{v_2, w\}$ is a dominating set of T with cardinality less than $\gamma(T)$. Hence v_1 is not adjacent to v_2 in T, and the desired result follows. □

Observation 5 *Each vertex of S_B' is a leaf.*

Proof: Let $u_1 \in S_B$, let $\operatorname{epn}(u_1, V_1^{(0)}) = \{v_1\}$ and suppose v_1 is not a leaf. Let $v_2 \in N(v_1) \backslash \{u_1\}$. Since S_B' is an independent set, it follows that $v_2 \in$

$N(S_A)$. Let u_2 be the vertex in S_A that is adjacent to v_2. Since $\deg_T u_1 \geq 2$, there exists a vertex $w \in N(u_1)\backslash\{v_1\}$. If $w \in V_1^{(0)}$, then $(V_1^{(0)}\backslash\{u_1, u_2\}) \cup \{v_2\}$ is a dominating set of T with cardinality less than $\gamma(T)$. Hence $w \in V(T)\backslash(V_1^{(0)} \cup S'_B) = N(S_A)$. Let u be the vertex in S_A adjacent to w. But then $(V_1^{(0)}\backslash\{u, u_1, u_2\}) \cup \{v_2, w\}$ is a dominating set of T with cardinality less than $\gamma(T)$. It follows that v_1 is a leaf. $\square$

Observation 6 *If $k \geq 2$, then $S_A = \emptyset$.*

Proof: Suppose $k \geq 2$, $S_A \neq \emptyset$, and let $y \in S_A$. Consider a vertex sequence $v_0, v_1, \ldots, v_{k-1}$ of T, with v_0 and v_1 distinct vertices in $N(y)$, which exists, since y is not a leaf in T. There exists a vertex $u_0 \in N(v_0) \cap V_1^{(0)}$ such that $f^{(1)} = \text{move}(f^{(0)}, u_0 \to v_0)$ is a safe guard function of T. If $u_0 \in S_B$, then $\text{epn}(u_0, V_1^{(0)})$ would be left undominated under $f^{(1)}$. Therefore $u_0 = y$, and $y, v_1 \in V_0^{(1)}$, with v_1 not adjacent to v_0. But there also exists a vertex $u_1 \in N(v_1) \cap V_1^{(1)}$ such that $f^{(2)} = \text{move}(f^{(1)}, u_1 \to v_1)$ is a safe guard function of T. It follows that $u_1 \in S_B$. But then $\text{epn}(u_1, V_1^{(0)})$ is left undominated under $f^{(2)}$. This contradiction shows that $S_A = \emptyset$. $\square$

If $S_A = \emptyset$, then it follows from Observation 2 that $V(T) = S_B \cup S'_B$. By Observation 5, each vertex in S'_B is a leaf. Thus, T is the corona of the tree $\langle S_B \rangle_T$. For the case $k = 1$ and $S_A \neq \emptyset$, let $L_A = N(S_A)$. It follows from Observations 1, 2, 3 and 5, and the fact that each vertex in $V_1^{(0)}$ has degree at least 2 in T, that $T \in \mathcal{T}$. The converse follows by utilisation of Propositions 6.18 and 6.19. $\blacksquare$

6.6.2 An Upper Bound on $\gamma_{\ell,k}$

An exploration of the relationship between $\gamma_{\ell_{\max},k}(G)$ and $\ell_{\max}\gamma(G)$ for a general graph G was initiated in §4.4. As mentioned, the necessary condition in Proposition 4.12 is not sufficient. Henning [18] was, however, able to provide a characterisation of forests F for which $\gamma_{k+1,k}(F) = (k+1)\gamma(F)$. The results obtained by him will henceforth be discussed, altered only to allow for maximally $\ell_{\max} = \min(\Delta, k+1)$ guards per vertex instead of $k+1$.

As mentioned by Henning [18], Gunther *et al.* [14] presented the following characterisation of trees with unique minimum dominating sets, stated here without proof.

Theorem 6.5 *A tree T of order at least 3 has a unique minimum dominating set if and only if T has a minimum dominating set S, such that $|\text{epn}(u, S)| \geq 2$ for every $u \in S$.* $\blacksquare$

For the rest of this subsection, only trees of order at least three will be considered, since obtaining values for the parameters for trees of order two or less, is a trivial matter. Also, it will be assumed throughout the rest of this section that $\ell_{\max} = \min(\Delta, k+1)$. An immediate consequence of the above stated theorem and Proposition 4.12 is contained in the following corollary.

Corollary 6.5 *If T is a tree for which $\gamma_{\ell_{\max},k}(T) = \ell_{\max}\gamma(T)$ for some $k \in \mathbb{N}$, then T has a unique minimum dominating set.*

Proof: Since $\gamma_{\ell_{\max},k}(T) = \ell_{\max}\gamma(T)$, it follows from Proposition 4.12 that there exists a minimum dominating set S, such that $|\text{epn}(u,S)| \geq \ell_{\max}$ for every $v \in S$. The result follows from Theorem 6.5. ■

Denote the set of all r–support vertices of T by $S_r(T)$ and note that $S_{r+1}(T) \subseteq S_r(T)$ for any $r \in \mathbb{N}$. The following lemma may be useful to prove the upcoming results.

Lemma 6.3 *If T is a tree with unique minimum dominating set S, then $S_{\ell_{\max}}(T) \subseteq S$ for any $k \in \mathbb{N}$.*

Proof: If $r \geq 2$, it holds that every r–support vertex of a tree T belongs to every minimum dominating set of T. The result follows directly from this observation. ■

The following two propositions are necessary to determine the above mentioned characterisation of forests F for which $\gamma_{\ell_{\max},k}(F) = \ell_{\max}\gamma(F)$.

Proposition 6.20 *Let T be a tree with unique minimum dominating set S and $k \in \mathbb{N}$. If $S = S_{\ell_{\max}}(T)$, then $\gamma_{\ell_{\max},k}(T) = \ell_{\max}\gamma(T)$.*

Proof: Let $f^{(0)}$ be an $(\ell_{\max}, k)$–SDF of T with minimum weight $\gamma_{\ell_{\max},k}(T)$. For each $u \in S$, let N_u consist of u and every leaf adjacent to u. Since $S = S_{\ell_{\max}}(T)$, u is adjacent to at least $\ell_{\max}$ leaves, and so $|N_u| \geq \ell_{\max} + 1$. Since the sets N_u are disjoint for different $u \in S$, it must hold that $f^{(0)}(N_u) \geq \ell_{\max}$ for every $u \in S$. It follows that $w(f^{(0)}) \geq \ell_{\max}\gamma(T)$, and the result follows by utilisation of Proposition 4.11. ■

Let $C_T(x)$ denote the set of all children of a vertex x in a rooted tree T. This notation will be used in the following proof, with the subscript dropped when reference to the tree in question is clear.

Proposition 6.21 *If T is a tree with unique minimum dominating set S, and if no vertex in S is an $\ell_{\max}$–support vertex, then $\gamma_{\ell_{\max},k}(T) < \ell_{\max}\gamma(T)$.*

Proof: The proof is by induction over trees T with the same value for $\gamma(T)$. Suppose T is a tree for which $\gamma(T) = 2$ and let $S = \{v_1, v_2\}$ be the unique minimum dominating set of T. Since $\gamma(T) = 2$ and T is connected and acyclic, the $v_1 - v_2$ path in T is of length at most 3. Furthermore, since S is unique and contains no $\ell_{\max}$–support vertices, the tree T consists only of the $v_1 - v_2$ path isomorphic to P_{a+1}, $1 \leq a \leq 3$, with n_1 and n_2 leaves joined to v_1 and v_2 respectively. Note that $2 \leq n_1, n_2 < \ell_{\max}$ if $a \in \{1,2\}$ and $1 \leq n_1, n_2 < \ell_{\max}$ if $a = 3$. The possible compositions of T are shown in Figure 6.5. If $a = 1$, the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$, with $V_{\ell_{\max}-1}^{(0)} = \{v_1, v_2\}$ and $V_0^{(0)} = V(T)\backslash\{v_1, v_2\}$ is an $(\ell_{\max}, k)$–SDF of T with weight $w(f^{(0)}) = 2(\ell_{\max} - 1) < \ell_{\max}\gamma(T)$. If $a \in \{2,3\}$, denote the internal vertex adjacent to v_1 by w. The safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$, with $V_{\ell_{\max}-1}^{(0)} = \{v_1, v_2\}$, $V_1^{(0)} = \{w\}$ and $V_0^{(0)} =$

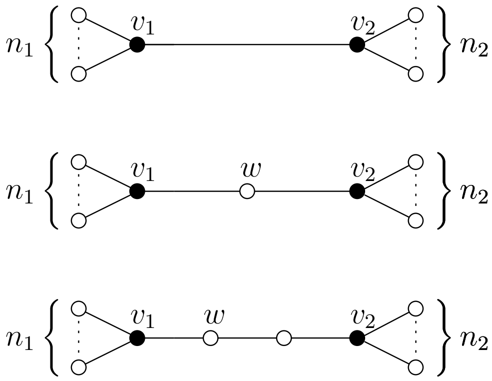


Figure 6.5: *Composition of trees T with unique minimum dominating set S (indicated as dark vertices) for which $\gamma(T) = 2$ and no vertex in S is an $\ell_{\max}$–support vertex.*

$V(T)\backslash\{v_1, v_2, w\}$ is an $(\ell_{\max}, k)$–SDF of T with weight $w(f^{(0)}) = 2\ell_{\max} - 1 < \ell_{\max}\gamma(T)$. So in all cases it holds that $\gamma_{\ell_{\max},k}(T) < \ell_{\max}\gamma(T)$.

Suppose that the result of the proposition is true for all trees T' with $\gamma(T') < t$, for some $t \geq 2$, that satisfy the hypothesis in the statement of the proposition. Let T be a tree with $\gamma(T) = t$ and with unique minimum dominating set S, such that no vertex in S is an $\ell_{\max}$–support vertex. Without loss of generality, let T be rooted at an end–vertex r of a longest path in T, of length p (say). Let w be a vertex at distance $p-2$ from r on a longest path in T starting at r, and let v be the child of w on this path. Let x denote the parent of w and let y denote the parent of x. Note that, as the proof progresses, it is only necessary to consider trees for which the inequality $\gamma_{\ell_{\max},k}(T) < \ell_{\max}\gamma(T)$ has not yet been established. To this end, a number of observations are provided, each time refining the number of trees still to be considered.

Observation 1 *It may be assumed that* $\deg_T v = \ell_{\max}$.

Proof: By Lemma 6.3 it holds that $S_{\ell_{\max}}(T) \subseteq S$. Since no vertex of S is an $\ell_{\max}$–support vertex, it follows that $S_{\ell_{\max}}(T) = \emptyset$. By Theorem 6.5, no leaf of T belongs to S, and so $v \in S$. Therefore v is adjacent to at most $\ell_{\max} - 1$ leaves. If $\deg_T v \leq \ell_{\max} - 1$, then $\mathrm{epn}(v, S)$ does not contain an independent set of $\ell_{\max}$ vertices. By Proposition 4.12 it then follows that $\gamma_{\ell_{\max},k}(T) < \ell_{\max}\gamma(T)$. Hence it may be assumed that $\deg_T v = \ell_{\max}$, since v is adjacent to a leaf at the end of a longest path from r. $\square$

Observation 2 *It may be assumed that $w \notin S$.*

Proof: Suppose $w \in S$. If $k = 1$, then $\ell_{\max} = 2$, and so $|\mathrm{epn}(v, S)| = 1$, which means that T does not have a unique minimum dominating set S — a contradiction. Therefore assume that $k \geq 2$ and consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$ of T, with $V_{\ell_{\max}}^{(0)} = S\backslash\{v\}$, $V_{\ell_{\max}-1}^{(0)} = \{v\}$ and $V_0^{(0)} = V(T)\backslash S$. Then $f^{(0)}$ is an $(\ell_{\max}, k)$–SDF of T with weight $w(f^{(0)}) = \ell_{\max}|S| - 1 = \ell_{\max}\gamma(T) - 1$. The desired result follows. $\square$

Observation 3 *It may be assumed that $x \notin S$.*

Proof: If $\text{epn}(v, S) < \ell_{\max}$, $\gamma_{\ell_{\max},k}(T) < \ell_{\max}\gamma(T)$ by Proposition 4.12. So it may be assumed that $\text{epn}(v, S) = N(v)$, since $\deg_T v = \ell_{\max}$, and hence $x \notin S$. $\square$

Observation 4 *It may be assumed that x is not a support vertex of T.*

Proof: If x is a support vertex of T, then it holds by Theorem 6.5 that $x \in S$, which contradicts the assumptions up to this point. $\square$

Observation 5 *It may be assumed that x does not have a child that is a support vertex of T.*

Proof: Suppose x has a child w' that is a support vertex of T. Then it follows, by Theorem 6.5, that $w' \in S$. If w' has a child v' that is a support vertex, then by the same argument as was used in Observation 2, it may be assumed that $w' \notin S$, which contradicts the assumptions proved valid thus far. Hence every child of w' is a leaf of T. By the same argument as was used in Observation 1, it may be assumed that $\deg_T w' = \ell_{\max}$. Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$ of T, with $V_{\ell_{\max}}^{(0)} = S \backslash \{v, w'\}$, $\{v, w'\} \subseteq V_{\ell_{\max}-1}^{(0)}$, $\{x\} \subseteq V_1^{(0)}$ and $V_0^{(0)} = V(T) \backslash (S \cup \{x\})$. Then $f^{(0)}$ is an $(\ell_{\max}, k)$–SDF of T with weight $w(f^{(0)}) = \ell_{\max}|S| - 1 = \ell_{\max}\gamma(T) - 1$. Thus, Observation 5 holds true. $\square$

Observation 6 *It may be assumed that* $\deg_T x = 2$.

Proof: Suppose $\deg_T x \geq 3$. Let $w' \in C(x)\backslash\{w\}$. Then w' is neither a leaf nor a support vertex of T, from Observations 4 and 5. Let v' be a child of w' and let u' be a child of v'. Since the $r - v$ path is a longest path in T starting from r, it follows that u' is a leaf. By the same argument as in Observation 1, it may be assumed that $\deg_T v' = \ell_{\max}$ and $v' \in S$. Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$ of T, with $V_{\ell_{\max}}^{(0)} = S\backslash\{v, v'\}$, $\{v, v'\} \subseteq V_{\ell_{\max}-1}^{(0)}$, $\{x\} \subseteq V_1^{(0)}$ and $V_0^{(0)} = V(T)\backslash(S \cup \{x\})$. Then $f^{(0)}$ is an $(\ell_{\max}, k)$–SDF of T with weight $w(f^{(0)}) = \ell_{\max}|S| - 1 = \ell_{\max}\gamma(T) - 1$. Therefore, it may be assumed that $\deg_T x = 2$. $\square$

Observation 7 *It may be assumed that* $\deg_T w = 2$.

Proof: Suppose $\deg_T w \geq 3$ and let $w' \in C(w)\backslash\{v\}$. By the same arguments as those used to prove Observations 4 and 5, it may be assumed that w' is neither a leaf nor a support vertex of T. So w' has a child, v' say, which has a child, u' say. But then there exists an $r - u'$ path of length longer than the path $r - v$ (which is the longest path from r in T). This contradiction provides the desired result. $\square$

To complete the proof, let $T' = T\backslash(C(v) \cup \{v, w, x\})$. It follows, by Observations 2, 3 and 6, that $y \in S$, and by Observation 7 it holds that $\deg_T w = 2$. Therefore $S\backslash\{v\}$ is a minimum dominating set of T', and so $\gamma(T') = |S| - 1 = \gamma(T) - 1$. Let h' be a minimum weight $(\ell_{\max}, k)$–SDF of T' and consider the safe guard function h of T, defined by $h(z) = h'(z)$ for all $z \in V(T')$, $h(x) = 1$, $h(v) = \ell_{\max} - 1$, $h(w) = 0$ and $h(u) = 0$ for each child u of v. Then h is an $(\ell_{\max}, k)$–SDF of T with weight $w(h) = w(h') + \ell_{\max} = \gamma_{\ell_{\max},k}(T') + \ell_{\max} \leq \ell_{\max}\gamma(T') + \ell_{\max} \leq \ell_{\max}\gamma(T)$. Suppose $\gamma_{\ell_{\max},k}(T') = \ell_{\max}\gamma(T')$. By Corollary 6.5 it holds that T' has a unique minimum dominating set, namely $S' = S\backslash\{v\}$. In particular, since $y \in S'$, y is not a leaf in T'. Hence, every leaf in T' is also a leaf in T. Therefore, since T has no $\ell_{\max}$–support vertex, neither does T'. Consequently, T' is a tree with $\gamma(T') < t$ and with a unique minimum dominating set S', such that no vertex in S' is an $\ell_{\max}$–support vertex of T. Applying the induction assumption to T', it follows that $\gamma_{\ell_{\max},k}(T') < \ell_{\max}\gamma(T')$, so that $\gamma_{\ell_{\max},k}(T) \leq w(h) < \ell_{\max}\gamma(T)$. ■

The following result for forests, follows directly from Proposition 6.21.

Corollary 6.6 *If F is a forest with unique minimum dominating set S, and if F has a component with no $\ell_{\max}$–support vertex, then $\gamma_{\ell_{\max},k}(F) < \ell_{\max}\gamma(F)$.*

Proof: Let F be the union of trees the $T_1, T_2, \ldots, T_n$, and suppose T_n has no $\ell_{\max}$–support vertex. By Proposition 6.21 it follows that $\gamma_{\ell_{\max},k}(T_n) < \ell_{\max}\gamma(T_n)$, so that the result follows from Lemma 4.1. ■

The characterisation of forests F for which $\gamma_{\ell_{\max},k}(F) = \ell_{\max}\gamma(G)$ is now presented, as established by Henning [18]. To this end, a family $\mathcal{F}$ of forests is defined as follows:

Let F be a forest with unique minimum dominating set S, such that each component of F contains an $\ell_{\max}$–support vertex. It follows from Lemma 6.3 that $S_{\ell_{\max}}(F) \subseteq S$. If $S_{\ell_{\max}}(F) = S$, then let $\tilde{F} = F$. If $S_{\ell_{\max}} \neq S$, a subforest $\tilde{F}$ of F is constructed as follows: Let $F_0 = F$ and for $i \in \mathbb{N}$, let $S_i = S \cap V(F_i)$. If every component of F_i contains an $\ell_{\max}$–support vertex and if $S_i \backslash S_{\ell_{\max}}(F_i) \neq \emptyset$, then let

$$F_{i+1} = F_i - \left(\bigcup_{v \in S_{\ell_{\max}}(F_i)} N[v] \backslash \left(S_i \backslash S_{\ell_{\max}}(F_i)\right)\right).$$

In other words, F_{i+1} is obtained from F_i by deleting all $\ell_{\max}$–support vertices and their neighbourhoods, except for other occupied vertices possibly included among these neighbourhoods. Since $\gamma(F)$ is finite, the sequence $F_0, F_1, \ldots$ will terminate in a forest F_t, for some $t \in \mathbb{N}$, such that F_t has a component with either no $\ell_{\max}$–support vertex, or $S_t = S_{\ell_{\max}}(F_t)$. Then, let $\tilde{F} = F_t$. For $i = 1, 2, \ldots, t$, the forest F_{i+1} is referred to as the *pruning* of F_i, while t is called the *number of prunings* of F. Note that $S_{i+1} = S_i \backslash S_{\ell_{\max}}(F_i)$ if $t > i \geq 0$. An example of the pruning of a forest is shown in Figure 6.6, with $\Delta = 4$, $k = 2$, and hence $\ell_{\max} = 3$. In each case, the unique minimum dominating set of the forest is indicated by the dark vertices.

The family $\mathcal{F}$ is defined to consist of all forests F, of which every component contains an $\ell_{\max}$–support vertex, that have a unique minimum dominating set S, such that $\tilde{F} = F_t$ and $S_t = S_{\ell_{\max}}(F_t)$. Note that, if $F \in \mathcal{F}$ and $\tilde{F} = F_t$, then each of the subforests $F_0, F_1, \ldots, F_t$ belongs to the family $\mathcal{F}$ as well, whereas for a forest $G \notin \mathcal{F}$, none of its prunings belongs to the family $\mathcal{F}$. The forest F_0 in Figure 6.6 belongs to the family $\mathcal{F}$, since pruning it results in a forest F_2 for which the unique minimum dominating set

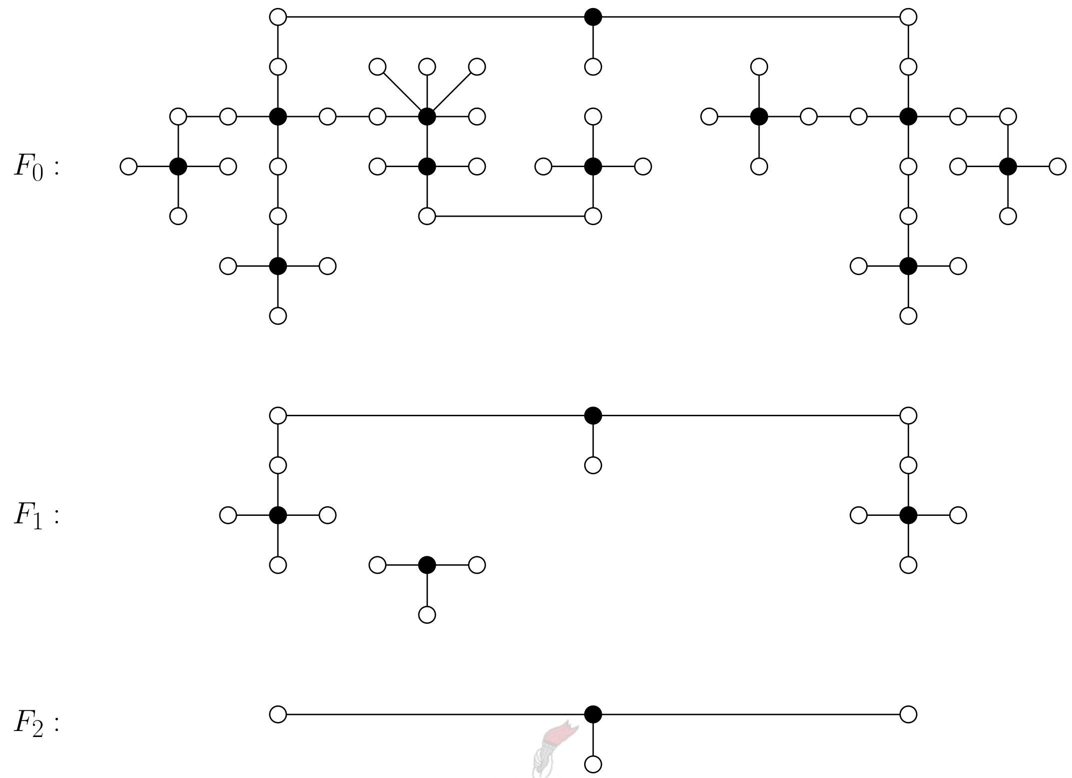


Figure 6.6: *An example of the pruning of the forest F_0, with $\Delta = 6$, $k = 2$ and hence $\ell_{\max} = 3$. The unique minimum dominating set in each case is indicated by the dark vertices.*

consists solely of $\ell_{\max}$–support vertices. The forest G_0 shown in Figure 6.7, however, does not belong to $\mathcal{F}$, since its pruning results in a forest G_2 of which a component does not contain an $\ell_{\max}$–support vertex.

Regarding the minimum dominating sets of these subforests, the following lemma was proved by Henning [18].

Lemma 6.4 *The set S_i is the unique minimum dominating set of the forest F_i, for any $i = 0, 1, \ldots, t$.*

Proof: The proof is by induction over i. If $i = 0$, then $S_0 = S$ and $F_0 = F$, so that S_0 is the unique minimum dominating set of F_0. Suppose that the set S_m is the unique minimum dominating set of F_m, $0 \leq m < t$. By construction, S_{m+1} is a dominating set of F_{m+1}, so that $\gamma(F_{m+1}) \leq |S_{m+1}|$. If $\gamma(F_{m+1}) < |S_{m+1}|$, then adding the set $S_{\ell_{\max}}(F_m)$ to any minimum dominating set of F_{m+1} produces a dominating set of F_m of cardinality

$$\begin{aligned} |S_{\ell_{\max}}(F_m)| + \gamma(F_{m+1}) &< |S_{\ell_{\max}}(F_m)| + |S_{m+1}| \\ &= |S_{\ell_{\max}}(F_m)| + |S_m \backslash S_{\ell_{\max}}(F_m)| \\ &= |S_m| \\ &= \gamma(F_m), \end{aligned}$$

which is a contradiction. Therefore $\gamma(F_{m+1}) = |S_{m+1}|$. If F_{m+1} has two distinct minimum dominating sets X and Y, then $X \cup S_{\ell_{\max}}(F_m)$ and $Y \cup S_{\ell_{\max}}(F_m)$ are two distinct mini-

G_0 :

G_1 :

G_2 :

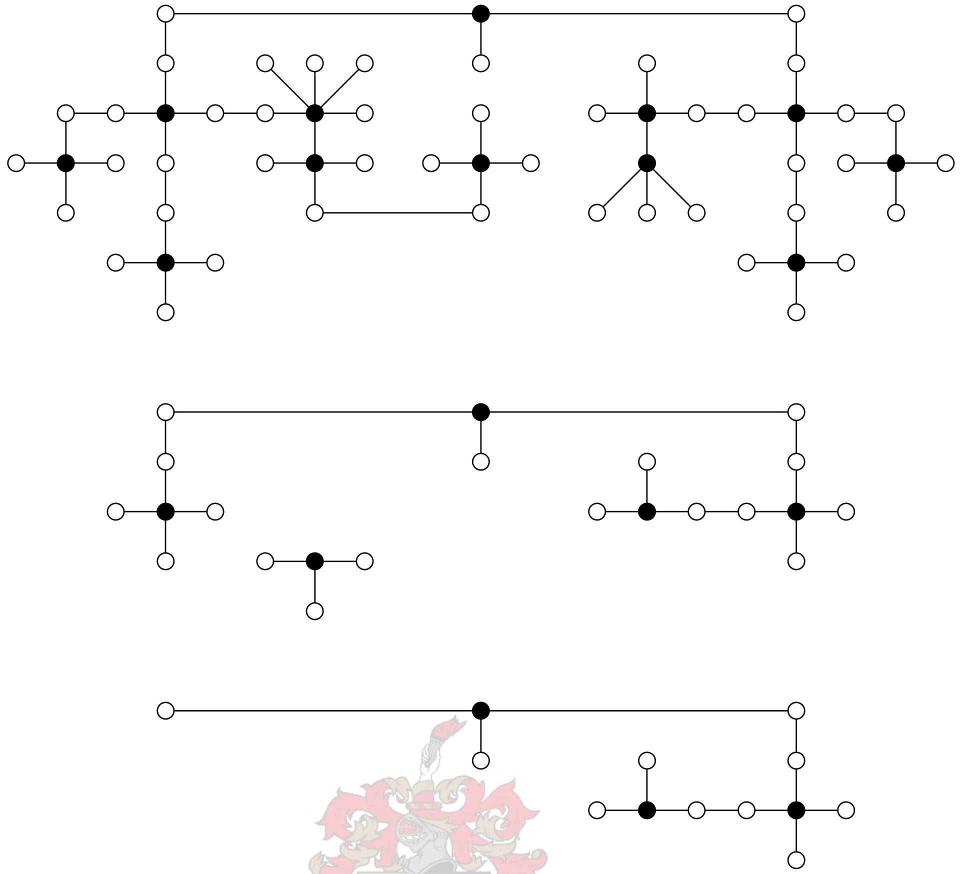


Figure 6.7: *An example of the pruning of the forest G_0, with $\Delta = 6$, $k = 2$, $\ell_{\max} = 3$, which does not belong to the family $\mathcal{F}$. The unique minimum dominating set in each case is indicated by the dark vertices.*

mum dominating sets of F_m, which is again a contradiction. Hence, S_{m+1} is the unique minimum dominating set of F_{m+1}. ■

The above mentioned characterisation is now presented as two propositions, with their combined result summarised in Theorem 6.6.

Proposition 6.22 *If $F \in \mathcal{F}$, then $\gamma_{\ell_{\max},k}(F) = \ell_{\max}\gamma(F)$.*

Proof: The proof is by induction over the number of prunings, t, of the forest F. Let S be the unique minimum dominating set of F. If $t = 0$, then $\tilde{F} = F$ and $S = S_{\ell_{\max}}(F)$. Thus every vertex in S is an $\ell_{\max}$–support vertex. It follows from Proposition 6.20 that $\gamma_{\ell_{\max},k}(F) = \ell_{\max}\gamma(F)$.

Now, suppose that all forests $F \in \mathcal{F}$, with $\tilde{F} = F_m$ for some $0 \leq m < t$ satisfy $\gamma_{\ell_{\max},k}(F) = \ell_{\max}\gamma(F)$. Let the pruning of $F \in \mathcal{F}$ terminate in $\tilde{F} = F_t$. Then it follows that $S_t = S_{\ell_{\max}}(F_t)$. Since $t \geq 1$ it holds that $S \backslash S_{\ell_{\max}}(F) \neq \emptyset$. Consider the forest $F_1 = F \backslash (\cup_{v \in S_{\ell_{\max}}(F)} N[v] \backslash S_1)$, resulting from the pruning of F. From Lemma 6.4 it follows that S_1 is the unique minimum dominating set of F_1. Since $F \in \mathcal{F}$, every component of F_1 has an $\ell_{\max}$–support vertex. Clearly $F_1 \in \mathcal{F}$ and $t-1$ prunings of the forest F_1 are needed to construct the forest $\tilde{F}_1$. Applying the induction hypothesis to F_1, it follows that $\gamma_{\ell_{\max},k}(F_1) = \ell_{\max}\gamma(F_1)$.

Let f_1 be a minimum weight $(\ell_{\max}, k)$–SDF of F_1, and consider the safe guard function f of F, defined by $f(v) = f_1(v)$ if $v \in V(F_1)$, $f(v) = \ell_{\max}$ if $v \in S_{\ell_{\max}}(F)$, and $f(v) = 0$ otherwise. Then f is an $(\ell_{\max}, k)$–SDF of F, with

$$\begin{aligned} w(f) &= w(f_1) + \ell_{\max}|S_{\ell_{\max}}(F)| \\ &= \gamma_{\ell_{\max},k}(F_1) + \ell_{\max}|S_{\ell_{\max}}(F)|. \end{aligned}$$

On the other hand, let g be a minimum weight $(\ell_{\max}, k)$–SDF of F. There will always exist a g for which $g(u) = \ell_{\max}$ for each $u \in S_{\ell_{\max}}(F)$ and $g(v) = 0$ for every leaf adjacent to u. Furthermore, there exists a g for which $g(v) = 0$ for any $v \in N[u]\backslash S$ and not a leaf, since otherwise, if $g(v) \neq 0$ always holds, it would mean that $v \in S$. Let g' be the restriction of g to F_1. Then g' is an $(\ell_{\max}, k)$–SDF of F_1, so that

$$\begin{aligned} \gamma_{\ell_{\max},k}(F_1) &\leq w(g') \\ &= w(g) - \ell_{\max}|S_{\ell_{\max}}(F)| \\ &= \gamma_{\ell_{\max},k}(F) - \ell_{\max}|S_{\ell_{\max}}(F)|. \end{aligned}$$

Consequently, $\gamma_{\ell_{\max},k}(F) = \gamma_{\ell_{\max},k}(F_1) + \ell_{\max}|S_{\ell_{\max}}(F)|$. Since S_1 is the unique minimum dominating set of F_1, it follows that

$$\begin{aligned} \gamma(F_1) &= |S_1| \\ &= |S| - |S_{\ell_{\max}}(F)| \\ &= \gamma(F) - |S_{\ell_{\max}}(F)|. \end{aligned}$$

It therefore follows that

$$\begin{aligned} \gamma_{\ell_{\max},k}(F) &= \gamma_{\ell_{\max},k}(F_1) + \ell_{\max}|S_{\ell_{\max}}(F)| \\ &= \ell_{\max}(\gamma(F_1) + |S_{\ell_{\max}}(F)|) \\ &= \ell_{\max}\gamma(F), \end{aligned}$$

since $\gamma_{\ell_{\max},k}(F_1) = \ell_{\max}\gamma(F_1)$. ■

The sufficient condition stated in the above proposition, is also necessary, as established in the following proposition.

Proposition 6.23 *If F is a forest for which $\gamma_{\ell_{\max},k}(F) = \ell_{\max}\gamma(F)$, then $F \in \mathcal{F}$.*

Proof: Suppose $F \notin \mathcal{F}$. If the forest F does not have a unique minimum dominating set, then it follows from Corollary 6.5 that $\gamma_{\ell_{\max},k}(F) < \ell_{\max}\gamma(F)$. Hence it may be assumed that F has a unique minimum dominating set S. If F has a component with no $\ell_{\max}$–support vertex, then by Corollary 6.6 it holds that $\gamma_{\ell_{\max},k}(F) < \ell_{\max}\gamma(F)$. Hence it may also be assumed that each component of F contains an $\ell_{\max}$–support vertex. Since $F \notin \mathcal{F}$, it follows that $\tilde{F} = F_t$, where F_t has a component with no $\ell_{\max}$–support vertices. Let g be a minimum weight $(\ell_{\max}, k)$–SDF of F_t. From Lemma 6.4 it follows that S_t is the unique minimum dominating set of F_t, and therefore $w(g) = \gamma_{\ell_{\max},k}(F_t) < \ell_{\max}\gamma(F_t)$, by Corollary 6.6. By construction it holds that $S\backslash S_t$ is a dominating set of $F - V(F_t)$. Consider the safe guard function f of F, defined by $f(v) = g(v)$ if $v \in V(F_t)$, $f(v) = \ell_{\max}$ if $v \in S\backslash S_t$, and $f(v) = 0$ otherwise. Then f is an $(\ell_{\max}, k)$–SDF of F, and so

$$\begin{aligned} \gamma_{\ell_{\max},k}(F) &\leq w(f) \\ &= w(g) + \ell_{\max}|S\backslash S_t| \\ &< \ell_{\max}\gamma(F_t) + \ell_{\max}(|S| - |S_t|) \\ &= \ell_{\max}|S_t| + \ell_{\max}(|S| - |S_t|) \\ &= \ell_{\max}|S| \\ &= \ell_{\max}\gamma(F). \end{aligned}$$

Hence, in all cases $\gamma_{\ell_{\max},k}(F) < \ell_{\max}\gamma(F)$. The result follows from the contra–positive of this. ■

The desired characterisation may now be summarised as follows.

Theorem 6.6 *For a forest F, $\gamma_{\ell_{\max},k}(F) = \ell_{\max}\gamma(F)$ if and only if $F \in \mathcal{F}$.*

Proof: This result follows immediately from Propositions 6.22 and 6.23. ■

Depending on the relationship between the maximum degree Δ of the forest in question, and the number of vertices to protect, k, it is possible to state more simply which forests belong to the family $\mathcal{F}$. The following corollary validates this claim.

Corollary 6.7 *Let F be a forest and $k \in \mathbb{N}$ such that $\Delta \leq k + 1$. Then $F \in \mathcal{F}$ if and only if F is the union of Δ–stars.*

Proof: Suppose $F \in \mathcal{F}$. Then it follows that every component of F contains an $\ell_{\max}$–support vertex, $\ell_{\max} = \min(\Delta, k + 1)$. Suppose F has a component T which is not a Δ–star. Then there exists a $v \in V(T) \cap S_{\ell_{\max}}(F)$ also adjacent to one internal vertex (i.e. not a leaf). But then $\deg_T v \geq \Delta + 1$, which is a contradiction. It is concluded that every component of F is a Δ–star. Also, if F is the union of Δ–stars, it holds that every component of F contains an $\ell_{\max}$–support vertex and $S = S_{\ell_{\max}}(F)$. Hence, $F \in \mathcal{F}$. ■

6.6.3 Trees of Special Structure

From Corollary 4.11 it is known that $\gamma_{\ell_{\max},k}(G) \leq \ell_{\max}\gamma(G)$ for any graph G, $k \in \mathbb{N}$, with $\ell_{\max} = \min(\Delta, k + 1)$. It is expected that this upper bound, although general, is usually not the best possible, and for some special graph classes, the smart parameter values may be determined explicitly. This is indeed the case for the subclass of trees called caterpillars, as defined in §2.1.5. First, a corollary deriving directly from the results obtained in §6.5, is stated.

Pectora roborant cultus recti

Corollary 6.8 *Consider the n–star $K_{1,n}$, $n \in \mathbb{N}$. For any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$,*

$$\gamma_{\ell,k}(K_{1,n}) = \begin{cases} k+1 & \text{if } 1 \leq k < n \text{ and } \ell \geq k+1 \\ n & \text{if } 1 \leq k < n \text{ and } \ell < k+1, \text{ or if } k \geq n. \end{cases}$$

Proof: The first case follows immediately from Proposition 6.15, while the second case follows immediately from Proposition 6.17. ■

The value of the smart finite order domination number for the caterpillar $C(q_1, q_2, \ldots, q_n)$, $n \in \mathbb{N}$, varies according to a relationship between the number of vertices to protect and the maximum number of guards allowed at a vertex, as established in the following two propositions. For ease of presentation in the proofs of these results, let L denote the set of all leaves of the caterpillar in question.

Proposition 6.24 *Let $k \in \mathbb{N}$ and $G \cong C(q_1, q_2, \ldots, q_n)$, with $n \in \mathbb{N}$. Consider the ordered sequence $(p_1, p_2, \ldots, p_n)$, such that $\{p_1, p_2, \ldots, p_n\} = \{q_1, q_2, \ldots, q_n\}$ and $0 < p_1 \leq \cdots \leq p_n$. If $k \geq p_n$, then let $i^* = n$. Else, if $k < p_1$, then let $i^* = 0$. Otherwise let $i^* \in \{1, 2, \ldots, n\}$ such that $p_{i^*} \leq k < p_{i^*+1}$. In all cases, $\gamma_{\ell_{\max},k}(G) = \sum_{j=1}^{i^*} p_j + \ell_{\max}(n - i^*)$, with $\ell_{\max} = \min(\Delta, k+1)$.*

Proof: Let $s_1, s_2, \ldots, s_n$ denote the support vertices of G, such that the path resulting from the removal of all leaves of G, be $\langle\{s_1, s_2, \ldots, s_n\}\rangle \cong P_n$, with s_j adjacent to p_j leaves in G, $j = 1, 2, \ldots, n$. Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$ of G, with $V_{\ell_{\max}}^{(0)} = \{s_j : \ j = i^*+1, \ldots, n\}$, $V_{p_j}^{(0)} = \{s_j : \ j = 1, 2, \ldots, i^*\}$, and $V_0^{(0)} = V(G) \backslash \{s_j : \ j = 1, 2, \ldots, n\}$. It follows that for any sequence $v_0, v_1, \ldots, v_{k-1}$ in $N[s_j]$, there exists a sequence $u_i \in N[v_i] \cap V(G) \backslash V_0^{(i)}$, $i = 0, 1, \ldots, k-1$, that protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$, for any $j \in \{1, 2, \ldots, n\}$. Clearly $f^{(0)}$ is an $(\ell_{\max}, k)$–SDF of G, so that

$$\gamma_{\ell_{\max},k}(G) \leq w(f^{(0)}) = \sum_{j=1}^{i^*} p_j + \ell_{\max}(n - i^*). \tag{6.31}$$

Suppose $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_{\ell_{\max}}^{(0)})$ is an $(\ell_{\max}, k)$–SDF of G with weight $w(f^{(0)}) = \gamma_{\ell_{\max},k}(G) < \sum_{j=1}^{i^*} p_j + \ell_{\max}(n - i^*)$. Then there exists a support vertex s_j such that either

(i) $f^{(0)}((N[s_j] \cap L) \cup \{s_j\}) < p_j$, with $1 \leq j \leq i^*$, or

(ii) $f^{(0)}((N[s_j] \cap L) \cup \{s_j\}) < \ell_{\max} = k+1$, with $i^*+1 \leq j \leq n$.

Suppose (i) holds. It follows that $s_j \notin V_0^{(0)}$, since otherwise there exists a leaf $v \in N(s_j) \cap L$ that is not dominated. Denote the leaves in $N(s_j) \cap V_0^{(0)}$ by $v_0, v_1, \ldots, v_{c-1}$, for some $c \in \mathbb{N}$ such that $1 \leq c \leq p_j \leq k$. Since $f^{(0)}(s_j) \leq c-1$, it follows that no sequence $u_i \in N[v_i] \cap V(G) \backslash V_0^{(i)}$, $i = 0, 1, \ldots, c-1$, can protect v_i, $i = 0, 1, \ldots, c-1$, under $f^{(0)}$, since v_{c-1} will necessarily be left undominated.

Suppose (ii) holds. Then $k < p_n$ and $\ell_{\max} = k+1$. Since $f^{(0)}((N[s_j] \cap L) \cup \{s_j\}) < \ell_{\max} = k+1 \leq p_j$, it again follows that $s_j \notin V_0^{(0)}$, since otherwise there exists a leaf $v \in N(s_j) \cap L$ that is not dominated under $f^{(0)}$. Since there are at least $k+1$ leaves adjacent to s_j, it follows by the same argument as in case (i) that there exists a vertex sequence of length $c \leq k$ in $N(s_j) \cap L$ that cannot be protected under $f^{(0)}$.

It follows that

$$\gamma_{\ell_{\max},k}(G) \geq \sum_{j=1}^{i^*} p_j + \ell_{\max}(n - i^*). \tag{6.32}$$

The desired result follows by a combination of (6.31) and (6.32). ■

The parameter value in the case where a limited number of guards are allowed per vertex, is considered next.

Proposition 6.25 *Let $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \ell_{\max} - 1\}$, with $\ell_{\max} = \min(\Delta, k+1)$ for the caterpillar $G \cong C(p_1, p_2, \ldots, p_n)$, $n \in \mathbb{N}$ and $p_j > 0$ for all $j = 1, 2, \ldots, n$. Then*

$$\gamma_{\ell,k}(G) = \sum_{j=1}^{n} p_j.$$

Proof: Let $s_1, s_2, \ldots, s_n$ denote the support vertices of G, such that the path resulting from the removal of all leaves of G, be $\langle\{s_1, s_2, \ldots, s_n\}\rangle \cong P_n$, with s_j adjacent to p_j leaves in G, $j = 1, 2, \ldots, n$. Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ of G, with $V_1^{(0)} = \cup_{j=1}^n N(s_j) \cap L$ and $V_0^{(0)} = V(G) \backslash L$. It follows that for any sequence $v_0, v_1, \ldots, v_{k-1}$ in $(N(s_j) \cap L) \cup \{s_j\}$, there exists a sequence $u_i \in N[v_i] \cap V(G) \backslash V_0^{(i)}$, $i = 0, 1, \ldots, k-1$, which protects v_i, $i = 0, 1, \ldots, k-1$, under $f^{(0)}$, for any $j \in \{1, 2, \ldots, n\}$. Clearly $f^{(0)}$ is an (ℓ, k)–SDF of G, so that

$$\gamma_{\ell,k}(G) \leq w(f^{(0)}) = \sum_{j=1}^{n} p_j. \tag{6.33}$$

Suppose $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, \ldots, V_\ell^{(0)})$ is an (ℓ, k)–SDF of G with weight $w(f^{(0)}) = \gamma_{\ell,k}(G) < \sum_{j=1}^n p_j$. Then there exists a support vertex s_j such that $f^{(0)}((N(s_j) \cap L) \cup \{s_j\}) < p_j$, since otherwise $w(f^{(0)}) \geq \sum_{j=1}^n p_j$. It follows that $s_j \notin V_0^{(0)}$, since otherwise there exists a leaf $v \in N(s_j) \cap L$ that is not dominated under $f^{(0)}$. Denote the leaves in $N(s_j) \cap V_0^{(0)}$ by $v_0, v_1, \ldots, v_{c-1}$, for some $1 \leq c \leq p_j$. Since $f^{(0)}(s_j) \leq c - 1$ and $f^{(0)}(s_j) \leq \ell \leq k$, it follows that no sequence $u_i \in N[v_i] \cap V(G) \backslash V_0^{(i)}$, $i = 0, 1, \ldots, c-1$, can protect v_i, $i = 0, 1, \ldots, c-1$, under $f^{(0)}$, because v_{c-1} will necessarily be left undominated. It follows that

$$\gamma_{\ell,k}(G) \geq \sum_{j=1}^{n} p_j. \tag{6.34}$$

The desired result now follows by a combination of (6.33) and (6.34). ■

When examining caterpillars, it seems useful to distinguish between support vertices, leaves, and internal vertices (non–leaves) that are not support vertices. To this end, some additional notation may prove useful. Consider the caterpillar $G \cong C(p_1, p_2, \ldots, p_n)$, $n \in \mathbb{N}$, and let $k \in \mathbb{N}$. Let $Y_>$ be the set of all support vertices s_j with p_j leaves, $j \in \{1, 2, \ldots, n\}$, for which $p_j > k$. Let $Y_\leq$ be the set of all support vertices s_j with p_j leaves, $j \in \{1, 2, \ldots, n\}$, for which $p_j \leq k$. Then $Y = Y_> \cup Y_\leq$ is the set of all support vertices of G. Let X_s be the set of all internal non–support vertices s_j such that $N(s_j) \cap Y_> \neq \emptyset$. Let X_u be the set of all internal non–support vertices s_j such that $N(s_j) \cap Y_> = \emptyset$. Then $X = X_s \cup X_u$ is the set of all internal non–support vertices of G. Again, let L be the set of all leaves of G. Summarising the previous two results on caterpillars, the following corollary shows that, if every internal vertex is a support vertex, then the edges between these vertices have no effect on the value of the smart finite order domination number.

Corollary 6.9 *Let $G \cong C(q_1, q_2, \ldots, q_n)$, $n \in \mathbb{N}$, with $X_s \cup X_u = \emptyset$, and let $\Delta = \Delta(G)$. Then*

$$\gamma_{\ell,k}(G) = \sum_{j=1}^{n} \gamma_{\ell,k}\left(K_{1,q_j}\right)$$

for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$.

Proof: Suppose $\ell = \ell_{\max} = \min(\Delta, k+1)$, and consider the ordered sequence $(p_1, p_2, \ldots, p_n)$, such that $\{p_1, p_2, \ldots, p_n\} = \{q_1, q_2, \ldots, q_n\}$ and $0 < p_1 \leq \cdots \leq p_n$. If $k \geq p_n$, then let $i^* = n$. Else, if $k < p_1$, then let $i^* = 0$. Otherwise let $i^* \in \{1, 2, \ldots, n\}$ such that $p_{i^*} \leq k < p_{i^*+1}$. Let $s_1, s_2, \ldots, s_n$ denote the support vertices of G, such that the path resulting from the removal of all leaves of G, be $\langle\{s_1, s_2, \ldots, s_n\}\rangle \cong P_n$, with s_j adjacent to p_j leaves in G, $j = 1, 2, \ldots, n$. It follows that

$$\gamma_{\ell,k}\left(\langle (N(s_j) \cap L) \cup \{s_j\}\rangle_G\right) = \gamma_{\ell,k}\left(K_{1,q_j}\right) = \begin{cases} q_j & \text{if } j = 1, 2, \ldots, i^* \\ k+1 & \text{if } j = i^*+1, \ldots, n. \end{cases}$$

It follows from Proposition 6.24 that $\gamma_{\ell,k}(G) = \sum_{j=1}^{n} \gamma_{\ell,k}\left(K_{1,q_j}\right)$.

Now suppose $\ell < \ell_{\max} = \min(\Delta, k+1)$. Then $\ell < k+1$, so that $\gamma_{\ell,k}(K_{1,q_j}) = q_j$ for $j = 1, 2, \ldots, n$. It follows from Proposition 6.25 that $\gamma_{\ell,k}(G) = \sum_{j=1}^{n} \gamma_{\ell,k}\left(K_{1,q_j}\right)$. In both cases, the desired result is obtained. ■

The above mentioned results pertain to caterpillars consisting only of leaves and support vertices, from the requirement that $X_s \cup X_u = \emptyset$. The remaining possible caterpillars, those which contain internal vertices that are not support vertices, are considered next.

Proposition 6.26 *Let $G \cong C(p_1, p_2, \ldots, p_n)$, $n \in \mathbb{N}$, with $X = X_u \cup X_s \neq \emptyset$, and let $\ell_{\max} = \min(\Delta, k+1)$. Then, for any $k \in \mathbb{N}$,*

(a) $\gamma_{\ell,k}(G) \leq \displaystyle\sum_{j=1;\ p_j \neq 0}^{n} \gamma_{\ell,k}(K_{1,p_j}) + \gamma_{\ell,k}\left(\langle X\rangle_G\right)$, *for any $\ell \in \{1, 2, \ldots, \ell_{\max} - 1\}$.*

(b) $\gamma_{\ell_{\max},k}(G) \leq \sum_{j=1;\ p_j\neq 0}^{n} \gamma_{\ell_{\max},k}(K_{1,p_j}) + \gamma_{\ell_{\max},k}(\langle X_u\rangle_G)$.

Proof: Let $s_1, s_2, \ldots, s_n$ denote the support vertices of G, such that the path resulting from the removal of all leaves of G, be $\langle\{s_1, s_2, \ldots, s_n\}\rangle \cong P_n$, with s_j adjacent to p_j leaves in G, $j = 1, 2, \ldots, n$. Note that the subgraphs $\langle X\rangle_G$ are all paths in G.

(a) For each $j \in \{1, 2, \ldots, n\}$ such that $s_j \in Y$, consider the graphs $\langle(N(s_j) \cap L) \cup \{s_j\}\rangle_G \cong K_{1,p_j}$, since $\ell < \ell_{\max} \leq k + 1$. The union, H say, of these subgraphs with $\langle X\rangle_G$ forms a spanning subgraph of G, so that

$$\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(H) = \sum_{j=1;\ p_j\neq 0}^{n} \gamma_{\ell,k}(K_{1,p_j}) + \gamma_{\ell,k}(\langle X\rangle_G), \tag{6.35}$$

by utilisation of Corollary 4.2 and Lemma 4.1.

(b) For each $i \in \{1, 2, \ldots, n\}$ such $s_i \in Y_{\leq}$, consider the graphs $\langle N(s_i)\backslash(Y \cup X) \cup \{s_i\}\rangle_G \cong K_{1,p_i}$. For each j such that $s_j \in Y_{>}$, consider the graphs $\langle N(s_j)\backslash Y \cup \{s_j\}\rangle_G$, which is isomorphic to either K_{1,p_j}, K_{1,p_j+1} or K_{1,p_j+2}. Since $s_j \in Y_{>}$, it follows that $k + 1 \leq p_j \leq \Delta$, so that $\gamma_{\ell_{\max},k}(K_{1,p_j}) = \gamma_{\ell_{\max},k}(K_{1,p_j+1}) = \gamma_{\ell_{\max},k}(K_{1,p_j+2}) = k + 1$ by Corollary 6.8. The union, H say, of all these subgraphs with $\langle X_u\rangle_G$ forms a spanning subgraph of G, so that

$$\gamma_{\ell_{\max},k}(G) \leq \gamma_{\ell_{\max},k}(H) = \sum_{j=1;\ p_j\neq 0}^{n} \gamma_{\ell_{\max},k}(K_{p_j}) + \gamma_{\ell_{\max},k}(\langle X_u\rangle_G), \tag{6.36}$$

by utilisation of Corollary 4.2 and Lemma 4.1. ■

Pectora roborant cultus recti

It is not certain whether the bounds in Proposition 6.26 are, in fact, sharp. The infinite order domination numbers of caterpillars still remain to be found, and are partly determined in the following result.

Proposition 6.27 *For the caterpillar $G \cong C(p_1, p_2, \ldots, p_n)$, $n \in \mathbb{N}$, with $X_u \cup X_s = \emptyset$, $\gamma_\infty(G) = \sum_{j=1}^{n} p_j$.*

Proof: From Propositions 5.1 and 6.25 it follows that $\gamma_\infty(G) \geq \sum_{j=1}^{n} p_j$. The safe guard function $f^{(0)} = (V(G)\backslash L, L)$ is clearly an ∞–SDF of G. Hence it follows that $\gamma_\infty(G) \leq w(f^{(0)}) = \sum_{j=1}^{n} p_j$, providing the desired result. ■

Though the value of the foolproof finite order domination number of caterpillars seems more difficult to determine, the infinite order parameter value follows immediately from Theorem 5.4, and is stated below.

Corollary 6.10 *For the caterpillar $G \cong C(p_1, p_2, \ldots, p_n)$, $n \in \mathbb{N}$,*

$$\gamma^*_\infty(G) = n + \sum_{j=1}^{n} p_j - 1.$$

Proof: Since the minimum degree of $C(p_1, p_2, \ldots, p_n)$ is $\delta = 1$ and the order of the graph is $n + \sum_{j=1}^{n} p_j$, the result follows directly from Theorem 5.4. ■

From Propositions 6.24, 6.25 and 6.27, the observation is made that $\gamma_{\ell,k}(G) = \gamma_\infty(G)$ for any $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$ if $k \geq \max(p_1, p_2, \ldots, p_n)$, for the caterpillar $G \cong C(p_1, p_2, \ldots, p_n)$ with $X_u \cup X_s = \emptyset$. Furthermore, $\gamma_{\ell,k}(G) < \gamma_\infty(G)$ if $k < \max(p_1, p_2, \ldots, p_n)$ and $\ell = \ell_{\max}$. A similar observation was made in §6.3 for the cycle C_n. An open problem resulting from these observations is to determine, for a general graph G, the smallest $k' \in \mathbb{N}$ such that $\gamma_{\ell,k}(G) = \gamma_\infty(G)$ for any $k \geq k'$. In the case of G being a caterpillar consisting only of leaves and support vertices, this question is answered. However, it is expected that extensive investigation of special cases be required to put forth a general result.

Another subclass of trees is the class of spiders. Depending on the composition of the spider, various upper bounds may be established. These bounds are expected to be sharp for the smart finite order domination numbers, but have not yet been proven to be so. A result on the smart finite order parameter for a path, is stated first.

Corollary 6.11 *For any $n, k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$, $0 \leq \gamma_{\ell,k}(P_{n+1}) - \gamma_{\ell,k}(P_n) \leq 1$.*

Proof: From Theorem 6.1 it is known that $\gamma_{\ell,k}(P_n) = \lceil \frac{2k+1}{4k+3} n \rceil$ and $\gamma_{\ell,k}(P_{n+1}) = \lceil \frac{2k+1}{4k+3} n + \frac{2k+1}{4k+3} \rceil$. Since $\frac{2k+1}{4k+3} < \frac{1}{2}$ for any $k \in \mathbb{N}$, the result follows. ■

The following three propositions provide upper bounds on the smart finite order domination number of a spider. Each proposition considers a variation in the composition of the graph, or the maximum number of guards that are allowed per vertex.

Proposition 6.28 *Let $S_{m\times n}$, $m, n \geq 3$, be a spider for which $\gamma_{\ell_{\max},k}(P_{n-2}) < \gamma_{\ell_{\max},k}(P_{n-1})$, with $\ell_{\max} = \min(\Delta, k+1)$, $\Delta = \Delta(S_{m\times n})$, and let $k \in \mathbb{N}$. Then*

$$\gamma_{\ell_{\max},k}(S_{m\times n}) \leq \ell_{\max} + \Delta\gamma_{\ell_{\max},k}(P_{n-2}).$$

Proof: Let v denote the central vertex of $S_{m\times n}$ and let $\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-1}, v\} \subset V(S_{m\times n})$, with v adjacent to $v_{j,n-1}$, induce a path isomorphic to P_n in $S_{m\times n}$, for $j = 1, 2, \ldots, m$. Then the subgraphs $\langle \{v\} \cup \{v_{j,n-1} : j = 1, 2, \ldots, m\} \rangle_{S_{m\times n}} \cong K_{1,m}$ and $\langle \{v_{j,1}, v_{j,2}, \ldots, v_{j,n-2}\} \rangle_{S_{m\times n}} \cong P_{n-2}$ for each $j \in \{1, 2, \ldots, m\}$. The union of these subgraphs, H say, forms a spanning subgraph of $S_{m\times n}$. Since $m = \Delta$ and $\ell_{\max} = \min(\Delta, k+1)$, it holds that $\gamma_{\ell_{\max},k}(K_{1,m}) = \ell_{\max}$, and hence it follows from Corollary 4.2, Lemma 4.1 and Corollary 6.8 that

$$\gamma_{\ell_{\max},k}(S_{m\times n}) \leq \gamma_{\ell_{\max},k}(H) = \ell_{\max} + \Delta\gamma_{\ell_{\max},k}(P_{n-2}).$$ ■

Since this upper bound only pertains to the smart order domination number in the case where no restriction is placed on the number of guards allowed per vertex, it is still necessary to examine the scenario where a limit is enforced.

Proposition 6.29 *Let $k \in \mathbb{N}$, $\ell \in \{1, 2, \ldots, \ell_{\max} - 1\}$ and $S_{m\times n}$, $m, n \geq 3$, be a spider for which $\gamma_{\ell,k}(P_{n-2}) < \gamma_{\ell,k}(P_{n-1})$, with $\ell_{\max} = \min(\Delta, k+1)$, and let $\Delta = \Delta(S_{m\times n})$. Then*

$$\gamma_{\ell,k}(S_{m\times n}) \leq \Delta + \Delta\gamma_{\ell,k}(P_{n-2}).$$

Proof: Let v denote the central vertex of $S_{m\times n}$ and $\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-1}, v\} \subset V(S_{m\times n})$, with v adjacent to $v_{j,n-1}$, induce a path isomorphic to P_n in $S_{m\times n}$, for $j = 1, 2, \ldots, m$. Then $\langle\{v\} \cup \{v_{j,n-1} : \ j = 1, 2, \ldots, \ell\}\rangle_{S_{m\times n}} \cong K_{1,\ell}$, $\langle\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-2}\}\rangle_{S_{m\times n}} \cong P_{n-2}$ for each $j \in \{1, 2, \ldots, \ell\}$ and $\langle\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-1}\}\rangle_{S_{m\times n}} \cong P_{n-1}$ for each $j \in \{\ell+1, \ell+2, \ldots, m\}$. The union of these subgraphs, H say, forms a spanning subgraph of $S_{m\times n}$. Since $\ell < \ell_{\max} = \min(\Delta, k+1)$, it holds that $\gamma_{\ell,k}(K_{1,\ell}) = \ell$, and hence it follows from Corollary 4.2, Lemma 4.1 and Corollary 6.8 that

$$\gamma_{\ell,k}(S_{m\times n}) \leq \gamma_{\ell,k}(H) = \ell + \ell\gamma_{\ell,k}(P_{n-2}) + (\Delta - \ell)\gamma_{\ell,k}(P_{n-1}) = \Delta + \Delta\gamma_{\ell,k}(P_{n-2}),$$

by utilisation of Corollary 6.11. ■

In both the previous results, it was required that the smart order domination number strictly increase when comparing the value for the paths P_{n-1} and P_{n-2}. The opposite of this condition is considered in the following proposition.

Proposition 6.30 *Let $k \in \mathbb{N}$, $\ell \in \{1, 2, \ldots, \ell_{\max}\}$ and $S_{m\times n}$, $m, n \geq 3$, be a spider for which $\gamma_{\ell,k}(P_{n-2}) = \gamma_{\ell,k}(P_{n-1})$, with $\ell_{\max} = \min(\Delta, k+1)$, and let $\Delta = \Delta(S_{m\times n})$. Then*

$$\gamma_{\ell,k}(S_{m\times n}) \leq 1 + \Delta\gamma_{\ell,k}(P_{n-2}).$$

Proof: Let v denote the central vertex of $S_{m\times n}$ and let $\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-1}, v\} \subset V(S_{m\times n})$, with v adjacent to $v_{j,n-1}$, induce a path isomorphic to P_n in $S_{m\times n}$, for $j = 1, 2, \ldots, m$. Consider the graph $\langle\{v\}\rangle_{S_{m\times n}}$, as well as $\langle\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-1}\}\rangle_{S_{m\times n}} \cong P_{n-1}$ for each $j \in \{1, 2, \ldots, m\}$. The union of these subgraphs, H say, forms a spanning subgraph of $S_{m\times n}$. Since $m = \Delta$, it follows from Corollary 4.2 and Lemma 4.1 that

$$\gamma_{\ell,k}(S_{m\times n}) \leq \gamma_{\ell,k}(H) = 1 + \Delta\gamma_{\ell,k}(P_{n-1}) = 1 + \Delta\gamma_{\ell,k}(P_{n-2}). \qquad ■$$

It may be possible to provide similar bounds for the smart finite order parameters in the case of wounded spiders, by using arguments similar to those in Propositions 6.28–6.30. However, the parameter value will, in each case, depend on the relationship between the various paths that make up the wounded spider in question. With an unlimited number of variations for the composition of such graphs, it is expected that such bounds will not be efficiently representable in general.

Explicit values for the smart infinite order domination number of spiders may, however, be provided. The following corollary proves useful in the proof of the upcoming result.

Corollary 6.12 *It holds that* $0 \leq \gamma_\infty(P_n) - \gamma_\infty(P_{n-1}) \leq 1$ *for any* $n \in \mathbb{N}$.

Proof: From Corollary 6.1 it is known that $\gamma_\infty(P_n) = \lceil \frac{n}{2} \rceil$ and $\gamma_\infty(P_{n-1}) = \lceil \frac{n-1}{2} \rceil$. The corollary follows immediately from this. ■

Utilising this result, the smart infinite order domination number of the spider is determined in the following result.

Proposition 6.31 *For the spider* $S_{m\times n}$, $m \geq 2$, $n \geq 3$,

$$\gamma_\infty(S_{m\times n}) = \begin{cases} \frac{1}{2}n\Delta & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1)\Delta + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let v denote the central vertex of $S_{m\times n}$ and let $\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-1}, v\} \subset V(S_{m\times n})$, with v adjacent to $v_{j,n-1}$, induce a path isomorphic to P_n in $S_{m\times n}$, for $j = 1, 2, \ldots, m$.

First, consider the case where n is even. From Proposition 5.1, Corollary 6.1 and Proposition 6.29 it follows that

$$\gamma_\infty(S_{m\times n}) \leq \Delta + \Delta\gamma_\infty(P_{n-2}) = \Delta + \Delta \left\lceil \frac{n-2}{2} \right\rceil = \frac{1}{2}n\Delta. \tag{6.37}$$

Suppose $f^{(0)}$ is an ∞–SDF of $S_{m\times n}$ of weight $w(f^{(0)}) = \gamma_\infty(S_{m\times n}) < \Delta + \Delta\gamma_\infty(P_{n-2})$. Then it follows that either

(i) $f^{(0)}(\{v\} \cup \{v_{j,n-1} : j = 1, 2, \ldots, m\}) < \Delta$, or

(ii) $f^{(0)}(\{v_{i,1}, v_{i,2}, \ldots, v_{i,n-2}\}) < \gamma_\infty(P_{n-2})$, for some $i \in \{1, 2, \ldots, m\}$.

Suppose (i) holds. It may be assumed, without loss of generality, that $f^{(0)}(\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-2}\}) = \gamma_\infty(P_{n-2})$ for $j = 1, 2, \ldots, m$. Then there exists a vertex sequence of length $1 \leq c < \Delta$ in $\{v\} \cup \{v_{j,n-1} : j = 1, 2, \ldots, m\}$, such that its protection under $f^{(0)}$ necessarily results in $f^{(c)}(\{v, v_{j^*,n-1}\}) = 0$ for some $j^* \in \{1, 2, \ldots, m\}$. Since n is even, it follows that $\gamma_\infty(P_{n-2}) < \gamma_\infty(P_{n-1})$. Therefore, there exists a vertex sequence of length $d \in \mathbb{N}$ in $\{v_{j^*,1}, v_{j^*,2}, \ldots, v_{j^*,n-2}\}$, such that its protection under $f^{(c)}$ necessarily results in a guard function $f^{(c+d)}$ that is not a safe guard function of $S_{m\times n}$.

Suppose (ii) holds. It may be assumed, without loss of generality, that $f^{(0)}(\{v\} \cup \{v_{j,n-1} : j = 1, 2, \ldots, m\}) = \Delta$ and $f^{(0)}(\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-2}\}) = \gamma_\infty(P_{n-2})$ for every $j \in \{1, 2, \ldots, m\}\backslash\{i\}$. Then there exists a vertex sequence of length $1 \leq c \leq \Delta$ in $\{v\} \cup \{v_{j,n-1} : j = 1, 2, \ldots, m\}$, such that its protection under $f^{(0)}$ necessarily results in $f^{(c)}(v) = 0$ and $f^{(c)}(v_{i,n-1}) = 1$. Since n is even, it follows from Corollary 6.12 that $\gamma_\infty(P_{n-1}) = \gamma_\infty(P_{n-2})+1$, so that $f^{(c)}(\{v_{i,1}, v_{i,2}, \ldots, v_{i,n-1}\}) < \gamma_\infty(P_{n-1})$. Therefore there exists a vertex sequence of length $d \in \mathbb{N}$ in $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n-1}\}$, such that its protection under $f^{(c)}$ necessarily results in a guard function $f^{(c+d)}$ that is not a safe guard function of $S_{m\times n}$.

It follows by contradiction that

$$\gamma_\infty(S_{m\times n}) \geq \Delta + \Delta\gamma_\infty(P_{n-2}) = \frac{1}{2}n\Delta, \tag{6.38}$$

so that the desired result is obtained by a combination of (6.37) and (6.38).

Now consider the case where n is odd. From Proposition 5.1, Corollary 6.1 and Proposition 6.30 it follows that

$$\gamma_\infty(S_{m\times n}) \leq 1 + \Delta\gamma_\infty(P_{n-2}) = 1 + \Delta\left\lceil\frac{n-2}{2}\right\rceil = \frac{1}{2}(n-1)\Delta + 1. \tag{6.39}$$

Suppose $f^{(0)}$ is an ∞–SDF of $S_{m\times n}$ of weight $w(f^{(0)}) = \gamma_\infty(S_{m\times n}) < 1 + \Delta\gamma_\infty(P_{n-1})$. Then either

(i) $f^{(0)}(v) = 0$, or

(ii) $f^{(0)}(\{v_{i,1}, v_{i,2}, \ldots, v_{i,n-1}\}) < \gamma_\infty(P_{n-1})$, for some $i \in \{1, 2, \ldots, m\}$.

Suppose (i) holds. It may be assumed, without loss of generality, that $w(f^{(0)}) = \gamma_\infty(S_{m\times n})-1$ and $f^{(0)}(\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-1}\}) = \gamma_\infty(P_{n-1})$ for $j = 1, 2, \ldots, m$. Then there exists a vertex sequence of length $0 \leq c \leq \Delta$ in $V(S_{m\times n})\backslash\{v\}$, such that its protection under $f^{(0)}$ necessarily results in $f^{(c)}(v_{j,n-1}) = 0$ for every $j \in \{1, 2, \ldots, m\}$, since $n-1$ is even. Since $f^{(c)}(v) = 0$ as well, $f^{(c)}$ is not a safe guard function of $S_{m\times n}$.

Suppose (ii) holds. It may be assumed, without loss of generality, that $f^{(0)}(v) = 1$ and $f^{(0)}(\{v_{j,1}, v_{j,2}, \ldots, v_{j,n-1}\}) = \gamma_\infty(P_{n-1})$ for every $j \in \{1, 2, \ldots, m\}\backslash\{i\}$. Then, again, there exists a vertex sequence of length $0 \leq c < \Delta$ in $V(S_{m\times n})\backslash\{v\}$, such that its protection under $f^{(0)}$ necessarily results in $f^{(c)}(v_{j,n-1}) = 0$ for every $j \in \{1, 2, \ldots, m\}\backslash\{i\}$, and $f^{(c)}(v) = 1$. Considering the subpath $\langle\{v_{i,1}, v_{i,2}, \ldots, v_{i,n-1}, v\}\rangle_{S_{m\times n}} \cong P_n$ of $S_{m\times n}$, it follows that $f^{(c)}(\{v_{i,1}, v_{i,2}, \ldots, v_{i,n-1}, v\}) < \gamma_\infty(P_{n-1}) + 1 = \gamma_\infty(P_n)$, since n is odd. Therefore there exists a vertex sequence of length $d \in \mathbb{N}$ in $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n-1}, v\}$, such that its protection under $f^{(c)}$ necessarily results in a guard function $f^{(c+d)}$ that is not a safe guard function of $S_{m\times n}$.

It follows by contradiction that

$$\gamma_\infty(S_{m\times n}) \geq 1 + \Delta\gamma_\infty(P_{n-1}) = \frac{1}{2}(n-1)\Delta + 1. \tag{6.40}$$

The desired result is obtained by a combination of (6.39) and (6.40). ■

Though results regarding the value of the foolproof finite order domination number of a spider seems more difficult to determine, the infinite order parameter value follows immediately from Theorem 5.4, and is stated below.

Corollary 6.13 *For the spider $S_{m\times n}$, $m \geq 2$, $n \geq 3$, $\gamma^*_\infty(S_{m\times n}) = m(n-1)$.*

Proof: Since the minimum degree of $S_{m\times n}$ is $\delta = 1$ and the order of the graph is $mn - m + 1$, the result follows directly from Theorem 5.4. ■

6.7 Hexagonal Graphs

As mentioned in §3.5, Burger *et al.* [3] noted the resemblance of higher order domination to a game of strategy. Since war games are typically played on boards consisting of hexagonal cells, the hexagonal graph $\mathcal{H}_{p,q}$ was considered, as defined in §2.1, and the infinite order domination parameters were explored for this graph, as shown below. Although the finite order domination numbers are yet undetermined for this class of graphs, the values and properties of these parameters may have great significance to the above–mentioned type of game.

Theorem 6.7 *For any integers $p, q \geq 2$,*

(a) $\gamma_\infty(\mathcal{H}_{p,q}) = \begin{cases} \frac{p}{2}\left\lceil\frac{2q}{3}\right\rceil & \text{if } p \text{ is even} \\ \frac{p-3}{2}\left\lceil\frac{2q}{3}\right\rceil + q + 1 & \text{if } p \text{ is odd.} \end{cases}$

(b) $\gamma^*_\infty(\mathcal{H}_{p,q}) = pq - 2.$

Proof: (a) Suppose p is even and consider the hexagonal graph $\mathcal{H}_{p,q}$ illustrated in Figure 6.8(a). For $i = 1, 2, \ldots, \frac{p}{2}$, define the triangular subgraphs

$$\overline{T}_{i,j} = \langle v_{2i-1,3j-2}, v_{2i,3j-2}, v_{2i-1,3j-1}\rangle$$

of $\mathcal{H}_{p,q}$ for $j = 1, 2, \ldots, \lceil\frac{q}{3}\rceil$, and

$$\underline{T}_{i,j} = \langle v_{2i-1,3j}, v_{2i,3j-1}, v_{2i,3j}\rangle$$

for $j = 1, 2, \ldots, \lceil\frac{q-1}{3}\rceil$. Adopt the convention that, if (for some triangle) one or two of the subscripts are out of range with respect to the vertex numbering of $\mathcal{H}_{p,q}$, it will appropriately be considered as a subgraph isomorphic to K_1 or K_2.

The subgraphs $\overline{T}_{i,j}$ and $\underline{T}_{i,j}$ constitutes a partition of the vertex set $V(\mathcal{H}_{p,q})$ into $\frac{p}{2}(\lceil\frac{q}{3}\rceil + \lceil\frac{q-1}{3}\rceil)$ independent cliques, as illustrated in Figure 6.8(b) (reproduced from [3]). It follows from Theorem 5.3 that

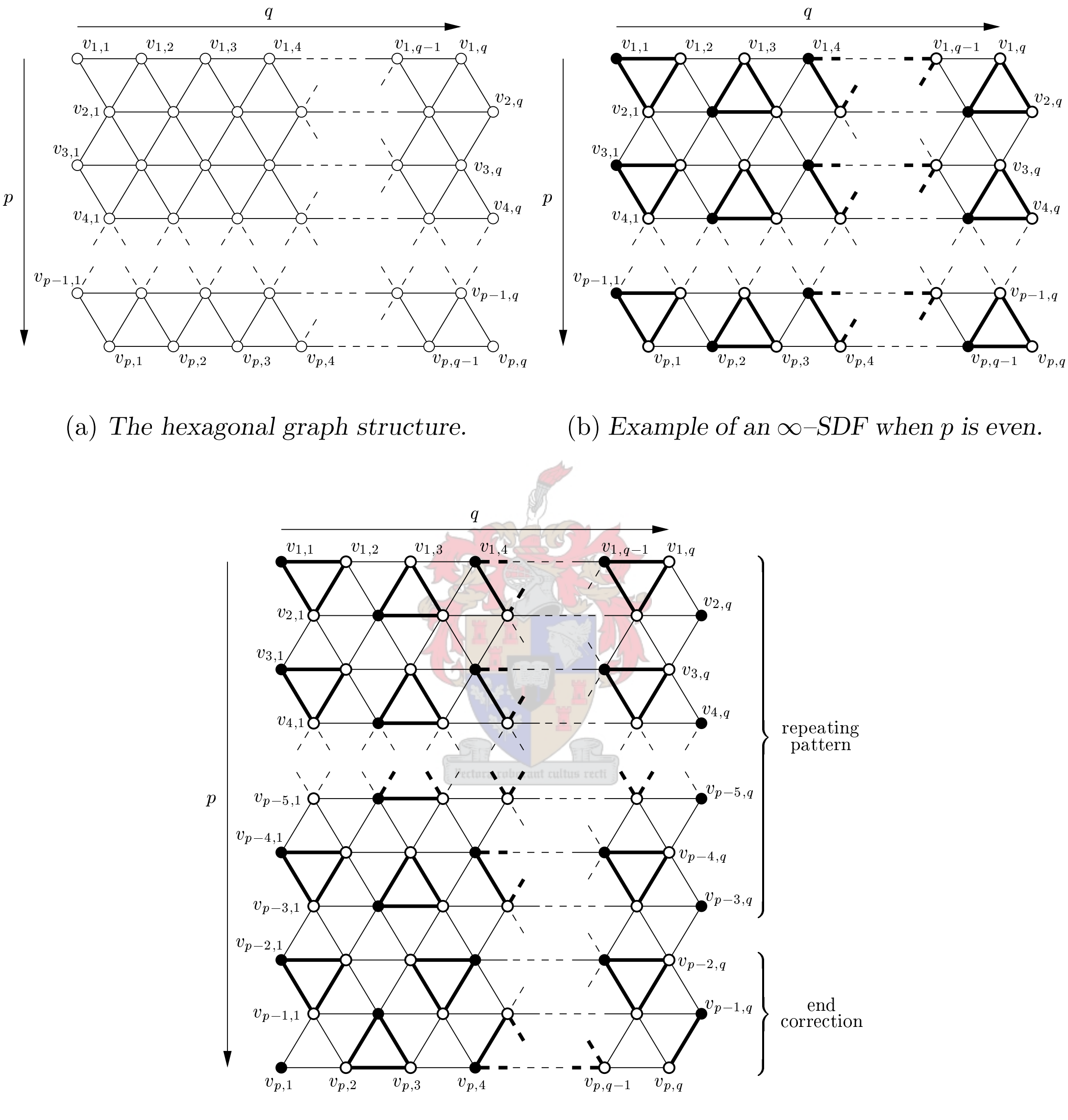
$$\gamma_\infty(\mathcal{H}_{p,q}) \leq \frac{p}{2}\left\lceil\frac{2q}{3}\right\rceil \quad \text{if } p \text{ is even.}$$

Since an independent set of cardinality $\frac{p}{2}\lceil\frac{2q}{3}\rceil$ can be constructed from the subgraphs $\overline{T}_{i,j}$ and $\underline{T}_{i,j}$, as indicated by the dark vertices in Figure 6.8(b) (reproduced from [3]), it follows by Proposition 5.5 that

$$\gamma_\infty(\mathcal{H}_{p,q}) \geq \frac{p}{2}\left\lceil\frac{2q}{3}\right\rceil \quad \text{if } p \text{ is even,}$$

thereby proving the first equality.

Pectora roborant cultus recti



(a) *The hexagonal graph structure.*

(b) *Example of an ∞–SDF when p is even.*

(c) *Example of an ∞–SDF when p is odd.*

Figure 6.8: *The hexagonal graph $\mathcal{H}_{p,q}$, as well as examples of ∞–SDF's depending on whether p is even or odd. (These figures are reproduced from [3].)*

Suppose p is odd and consider the subgraphs $\overline{T}_{i,j}$ and $\underline{T}_{i,j}$, for $i = 1, 2 \ldots, \frac{p-3}{2}$ and j as stated above, as well as the subgraphs

$$\overline{S}_{i,j} = \langle v_{p-2,2j-1}, v_{p-1,2j-1}, v_{p-2,2j} \rangle$$

for $j = 1, 2, \ldots, \lceil \frac{q}{2} \rceil$, and

$$\underline{S}_{i,j} = \langle v_{p-1,2j}, v_{p,2j}, v_{p,2j+1} \rangle$$

for $j = 0, 1, \ldots, \lfloor \frac{q}{2} \rfloor$. These subgraphs form a partition of $V(\mathcal{H}_{p,q})$ into $\frac{p-3}{2}(\lceil \frac{q}{3} \rceil + \lceil \frac{q-1}{3} \rceil) + q + 1$ independent cliques (as illustrated in Figure 6.8(c), reproduced from [3]), so that

$$\gamma_\infty(\mathcal{H}_{p,q}) \leq \frac{p-3}{2} \left\lceil \frac{2q}{3} \right\rceil + q + 1 \quad \text{if } p \text{ is odd.}$$

Again an independent set of cardinality $\frac{p-3}{2}\lceil \frac{2q}{3} \rceil + q + 1$ can be constructed from these subgraphs, as indicated by the dark vertices in Figure 6.8(c) (reproduced from [3]), so that by Proposition 5.5,

$$\gamma_\infty(\mathcal{H}_{p,q}) \geq \frac{p-3}{2} \left\lceil \frac{2q}{3} \right\rceil + q + 1 \quad \text{if } p \text{ is odd,}$$

which proves the second equality and therefore the desired result.

(b) Since the order of $\mathcal{H}_{p,q}$ is pq and the minimum degree is $\delta = 2$, the result follows directly from Theorem 5.4. ■

6.8 Chapter Summary

This chapter contained investigations of the higher order domination numbers for some special graph classes. As suspected, obtaining the higher order domination numbers for the complete graphs (discussed in §6.1) was a trivial matter. Burger *et al.* [2, 3] obtained most of the parameter values when considering paths and cycles (discussed in §6.2 and §6.3). The only unresolved issue was whether the foolproof parameter differs with a varying the number of guards allowed per vertex. Cartesian products of complete graphs, paths and cycles respectively, were considered in §6.4 and the infinite order domination parameters, as originally determined in [3], discussed. The lower bound on the smart parameter for the cartesian product of two cycles was improved upon, though its exact value is still unknown. In §6.5, both the smart and foolproof higher order domination numbers were resolved for the class of complete multipartite graphs, extending the result by Benecke *et al.* [1]. This was achieved by considering the relationship between the cardinalities of the partite sets and the number of attacks to protect against. Results obtained by Henning [17] for the class of trees, were discussed in §6.6, wherein two notable characterisations are provided for the smart finite order domination number. Additional to this discussion, the special class of caterpillars and spiders were also investigated in this section. Depending on the composition of these graphs, the value of the smart higher order domination number was determined. Lastly, the resemblance of higher order domination to a game of strategy was noted by Burger *et al.* [3]. Since war games are typically played on boards consisting of hexagonal cells, the hexagonal graph was investigated in §6.7, and the infinite higher order domination parameters explored for this graph.

Chapter 7

Conclusion

A summary of the work contained in this thesis is provided in §7.1. As conclusion to the thesis, a number of open problems touched upon throughout the exposition are summarised and highlighted for further work in §7.2. Finally, the definitions of the protection parameters provided in Chapters 4 and 5 may be generalised in various ways. Some possible generalisations are discussed informally in §7.3, suggesting a possibly sensible framework for future research on the topic of higher order domination.

7.1 Thesis Summary

The first recorded defence–location problem, that of Constantine the Great (274–337 A.D.), was used in Chapter 1 to introduce the concepts leading to the notion of higher order domination. An informal description of the various previous definitions of protection scenarios was also provided. Chapter 2 contains the basic graph and complexity theoretic concepts required to facilitate an understanding of the rest of the thesis. By way of establishing the background leading to the current framework for higher order domination, a comprehensive review of the combinatorial literature on Roman domination, weak Roman domination, secure domination and higher order domination was provided in Chapter 3. These notions were defined formally using the unifying notation consistent with the main body of this thesis.

In Chapter 4, a framework for finite order domination was introduced, generalising the definitions provided by Burger *et al.* [2], in the sense of allowing an arbitrary number of guards to be stationed at a vertex. The concepts of smart and foolproof domination were again distinguished. In the case of smart higher order domination, only the existence of a protection strategy is required, while in the foolproof case, it is required that *any* possible strategy should protect the graph. The numerous results obtained in [2] were shown to apply to the generalised parameters, by examining their growth properties and the effect of edge–removals in the graph, thereby meeting the goal set out in thesis Objective I, stated in §1.3. Also, certain additional novel results were established, indicated throughout the thesis by an asterisk, thereby partly achieving thesis Objective II. Further results on the smart parameters, obtained by Henning [18], were generalised to allow for a limited number of guards per vertex. This discussion, contained in §4.4, provided an introductory exploration of the question of when it is beneficial (in the sense of a decreased parameter value) to place multiple guards at a vertex. As a conclusion to this chapter, the complexity of determining a special case of the smart finite order domination number was shown to be NP–complete, as originally shown by Henning [18]. This verifies the expected computational difficulty involved when investigating the parameter values.

Chapter 5 concerned so–called perpetual security in a graph. Two definitions, similar to those provided by Burger *et al.* [3], are introduced, catering for smart and foolproof infinite order domination. The existence of these infinite order parameters was confirmed and it was shown that there are, in fact, only two infinite order domination numbers, since it makes no difference how many guards are allowed per vertex when the graph is attacked perpetually. Confirmation of this fact, in the generalised framework introduced in this thesis, adheres to the requirement of thesis Objective I, stated in §1.3. An exact value for the foolproof infinite order domination number was determined explicitly by Burger *et al.* [3], while its smart counterpart proved to be significantly more difficult to examine, as discussed in §5.5. General bounds on this parameter were obtained in [3], and it was confirmed that these bounds were simultaneously sharp only for perfect graphs. Although further investigation is required for these general bounds, it was noted that much tighter bounds may exist when considering special classes of graphs.

Such considerations were conducted in Chapter 6. As expected, obtaining the higher order domination numbers for the complete graphs (discussed in §6.1) was a trivial matter. Burger *et al.* [2, 3] obtained most of the parameter values when considering paths and cycles (§6.2 and §6.3). The only unresolved issue was whether the foolproof parameter for these graph structures differs with a varying the number of guards allowed per vertex. Cartesian products of complete graphs, paths and cycles respectively (§6.4), were considered and the infinite order domination parameters, as originally determined in [3], were reviewed. The lower bound on the smart parameter for the cartesian product of two cycles was also improved upon, although its exact value is still not known. In §6.5, both the smart and foolproof higher order domination numbers were resolved for the class of complete multipartite graphs, extending the result by Benecke *et al.* [1]. This was achieved by considering the relationship between the cardinalities of the partite sets and the number of attacks to protect against. The results obtained in this section achieved thesis Objective II to some extent, stated in §1.3. Results obtained by Henning [17] for the class of trees, were discussed in §6.6, wherein two notable characterisations were provided for the smart finite order domination number. The first of these concerned a characterisation of trees for which the finite order domination number is equal to the classical domination number, γ. The second provided a characterisation of forests for which the finite order domination number is equal to $(k+1)\gamma$, for the case where no restriction is placed on the number of guards allowed at a vertex. Additional to this discussion, the special graph classes of caterpillars and spiders were also investigated in this section. Depending on the composition of these graphs, the value of the smart higher order domination number was determined, adding towards the achievement of thesis Objective II. Finally, the resemblance of higher order domination to a game of strategy was noted by Burger *et al.* [3]. Since war games are typically played on boards consisting of hexagonal cells, the hexagonal graph was investigated in §6.7, and the infinite higher order domination parameters were explored for this graph.

7.2 Further Work

Throughout this thesis, a number of open problems have (directly or indirectly) been touched upon. These problems are summarised in this section, by way of a number of open binary questions, as required in thesis Objective II, stated in §1.3.

As mentioned in §4.3, it was discovered by Burger *et al.* [2] that the decomposition of a graph by means of an edge–removal, potentially increases the smart higher order domination number. This was stated in Lemma 4.5, namely that $\gamma_{\ell,k}(G) \leq \gamma_{\ell,k}(G-e)$ for any graph G, any edge $e \in E(G)$, any $k \in \mathbb{N}$ and any $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$. This led to the observation that the smart parameter of a graph may only possibly increase when considering any spanning subgraph thereof, as stated in Proposition 4.2. The question that arises from this observation is whether an equivalent result exists for the foolproof parameters, but counter examples were provided in §4.3. Examples of graphs for which such an equivalent result indeed holds, and others for which the opposite inequality holds, were found in §4.3. It is expected that certain graph properties will result in an increase in the foolproof parameter value under an edge–removal, while other properties may cause the opposite to happen, leading to the following two questions.

Question 7.1 *Is it possible to characterise graphs G for which*

(a) $\gamma^*_{\ell,k}(G) \leq \gamma^*_{\ell,k}(G-e)$

(b) $\gamma^*_{\ell,k}(G) > \gamma^*_{\ell,k}(G-e)$

for any edge $e \in E(G)$, and any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, k+1\}$? ■

Question 7.2 *Let $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, k+1\}$. Does there exist a graph G such that $\gamma^*_{\ell,k}(G) \leq \gamma^*_{\ell,k}(G-e_1)$ for some edge $e_1 \in E(G)$, but $\gamma^*_{\ell,k}(G) > \gamma^*_{\ell,k}(G-e_2)$ for some other edge $e_2 \in E(G)$? If so, can such graphs be characterised?* ■

The problem of determining exactly when it is beneficial (in the sense of a decreased parameter value) to station more guards at a vertex, is undoubtedly an intriguing one. It is expected that the mystery surrounding especially the smart finite order domination parameters may be cleared up to some extent by resolving this problem. An obvious lower bound for the higher order domination numbers are stated in Proposition 4.7. As a preliminary exploration, a comparison between the higher order domination parameters and the domination number was conducted in §4.4. Generalising results of Henning [18] to allow for a variable number of guards maximally allowed per vertex, it was shown in Proposition 4.8 that, if the smart finite order parameter is equal to the domination number, the number of guards allowed per vertex does not affect this minimum value. Furthermore, a sufficient condition on graphs G for which $\gamma_{\ell,k}(G) = \gamma(G)$ was provided. For the case $k = 1$, Henning and Hedetniemi [17] established a characterisation of this equality, as stated in Theorem 4.2, but the following question still remains open.

Question 7.3 *Is it possible to characterise graphs G for which $\gamma_{\ell,k}(G) = \gamma(G)$, similar to Theorem 4.2 for the case $k = 1$, but also holding true for any $k > 1$?* ■

Henning [18] was, however, able to obtain a characterisation of trees T for which $\gamma_{\ell,k}(T) = \gamma(T)$ for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, \min(\Delta, k+1)\}$. This characterisation was described in §6.6.1. Regarding the foolproof finite order domination numbers, no results comparing $\gamma^*_{\ell,k}$ with γ have been obtained. Hence, the following question arises.

Question 7.4 *Other than the complete graph, does there exist a graph G satisfying $\gamma^*_{\ell,k}(G) = \gamma(G)$ for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, k+1\}$?* ■

An upper bound relating the smart finite order domination numbers to the classical domination number was also provided in §4.4. A necessary, but not sufficient, condition on graphs G for which $\gamma_{\ell_{\max},k}(G) = \ell_{\max}\gamma(G)$ for any $k \in \mathbb{N}$, with $\ell_{\max} = \min(\Delta, k+1)$, was provided by Henning [18], stated in Proposition 4.12. The following question summarises the immediate open problems emanating from that discussion. Only for forests F have a characterisation been obtained (by Henning [18]) for when $\gamma_{\ell_{\max},k}(F) = \ell_{\max}\gamma(F)$, as discussed in §6.6.2.

Question 7.5 *Is it possible to characterise graphs G for which $\gamma_{\ell_{\max},k}(G) = \ell_{\max}\gamma(G)$ for any $k \in \mathbb{N}$? For such graphs G,*

(a) does there exist an $\ell' < \ell_{\max}$ such that $\gamma_{\ell,k}(G) = \ell'\gamma(G)$, or

(b) is $\gamma_{\ell,k}(G) = \ell_{\max}\gamma(G)$ for all $\ell \in \{1, 2, \ldots, \ell_{\max}\}$? ■

Again, for the foolproof finite order domination numbers, no results comparing $\gamma^*_{\ell,k}$ with γ have been obtained. The following question is therefore posed.

Question 7.6 *For a graph G, can an upper bound on $\gamma^*_{\ell_{\max},k}(G)$ be obtained in terms of $\gamma(G)$?* ■

Proposition 5.1 states that both the smart and foolproof finite order domination numbers are bounded from above by their infinite order counterparts, respectively. Although this is an intuitive result, it is not known exactly when the finite order parameter is equal to the infinite parameter for the first time as the number of attacks, k, increases. Obviously this equality occurs, since the finite order parameters are increasing in k, by Proposition 4.3. In the discussion on caterpillars (§6.6.3), the occurrence of this equality was determined in the case of the smart domination parameters. This occurrence was also discussed in 6.3 for the special graph classes of paths and cycles, as initially determined by Burger *et al.* [2]. The properties required to characterise this equality occurrence in general is, however, unclear. The following question summarises the resulting open problems.

Question 7.7

(a) For any graph G, is it possible to determine a $k'(G) \in \mathbb{N}$, such that $\gamma_{\ell,k'}(G) = \gamma_\infty(G)$ for every $k \geq k'$?

(b) *Does the value of k' perhaps depend on ℓ?*

(c) *Can similar integers be found for the foolproof case?*

(d) *Are they different from those in the smart case?* ■

As discussed in §5.4, the foolproof infinite order parameter $\gamma^*_\infty(G)$ has been determined explicitly for any graph G by Burger *et al.* [3]. This is, however, not the case for the smart infinite order parameter, as mentioned in §5.5. The general bounds $\beta(G) \leq \gamma_\infty(G) \leq \chi(\overline{G})$ were obtained by Burger *et al.* [3]. It is known that $\beta(G) = \gamma_\infty(G) = \chi(\overline{G})$ if G is a perfect graph, although this condition is not necessary. For the case where $\beta(G) < \chi(\overline{G})$ it is not certain for which graphs either one of the bounds is sharp. This provides the following open problem.

Question 7.8 *Is it possible to obtain a characterisation of graphs G for which*

(a) $\beta(G) = \gamma_\infty(G) < \chi(\overline{G})$?

(b) $\beta(G) < \gamma_\infty(G) = \chi(\overline{G})$? ■

Goddard *et al.* [12] determined that, for graphs G satisfying $\beta(G) = 2$, it holds that $\gamma_\infty(G) \leq 3$, as raised in Proposition 5.8. An open problem stated in [12], asked whether an integer $s_3 < \chi(\overline{G})$ exists, such that for any graph G with $\beta(G) = 3$, it holds that $\gamma_\infty(G) \leq s_3$. This question remains unanswered, and may be generalised as follows.

Question 7.9 *Do there exist integers $s_i < \chi(\overline{G})$, $i \geq 3$, such that for any graph G with $\beta(G) = i$, it holds that $i \leq \gamma_\infty(G) \leq s_i$?* ■

When considering special classes of graphs, it is expected that much better results concerning the higher order domination numbers may be possible than for general graphs. A number of special graph classes were discussed in Chapter 6, and although that chapter serves only as an initial exploration of the parameters, some unanswered questions already arose from this initial investigation. The smart finite order parameters for paths and cycles were completely determined by Burger *et al.* [2] and found to be equal for the respective graph classes, as stated in Theorems 6.1(a) and 6.2(a). However, in the case of the foolproof finite order parameters, only the case $\ell = 1$ has been resolved, as stated in Theorems 6.1(b) and 6.2(b). Although it is conjectured that the parameter value does not differ with an increase in ℓ, this assertion has not been proved.

Question 7.10 *Is it true that $\gamma^*_{\ell,k}(P_n) = \gamma^*_{1,k}(P_n) = \gamma^*_{\ell,k}(C_n)$ for any $n, k \in \mathbb{N}$ and $\ell \in \{1, 2\}$?* ■

A class of graphs related to cycles, is the class of wheels. In Propositions 6.4 and 6.5, upper bounds on the smart finite order parameter for this class were provided. It is expected that these upper bounds are sharp, but the author was unable to prove this.

Question 7.11 *For the wheel W_n and any $k \in \mathbb{N}$, is it true that*

(a) $\gamma_{\ell,k}(W_n) = \lceil \frac{2k+1}{4k+3} n \rceil$ *for any* $\ell \in \{1, 2, \ldots, \ell_{\max} - 1\}$, *and*

(b) $\gamma_{\ell_{\max},k}(W_n) = \min(k+1, \lceil \frac{2k+1}{4k+3} n \rceil)$,

with $\ell_{\max} = \min(\Delta, k+1)$? ■

In §6.4, the cartesian products of complete graphs, paths and cycles were considered. Bounds on the smart infinite order domination number of the cartesian product of cycles were provided by Burger *et al.* [3]. The lower bound was, however, improved upon in this thesis, as stated in Proposition 6.12. It was conjectured by Burger *et al.* [3] that the upper bound was, in fact, sharp if the product consisted of cycles of length greater than three. This conjecture is still unresolved, as stated in the following question.

Question 7.12 *Is* $\gamma_\infty(C_p \times C_q) = \lceil \frac{pq}{2} \rceil$ *for all* $p, q \geq 4$? ■

In §6.6.3, the special class of trees, called caterpillars, was considered, and certain results relating to the smart higher order domination numbers were established for this class. If the caterpillar consists only of leaves and support vertices, the parameter values were determined in Propositions 6.24 and 6.25. If this is not the case, in other words if there exists a vertex which is neither a leaf nor a support vertex, the parameter values are still unknown. However, upper bounds, which are expected to be sharp, were provided in Proposition 6.26. To this end, some additional notation is stated here once more. Consider the caterpillar $G \cong C(p_1, p_2, \ldots, p_n)$, $n \in \mathbb{N}$, and let $k \in \mathbb{N}$. Let $Y_>$ be the set of all support vertices s_j with leaves p_j, $j \in \{1, 2, \ldots, n\}$, for which $p_j > k$. Let $Y_\leq$ be the set of all support vertices s_j with leaves p_j, $j \in \{1, 2, \ldots, n\}$, for which $p_j \leq k$. Then $Y = Y_> \cup Y_\leq$ is the set of all support vertices of G. Let X_s be the set of all internal non–support vertices s_j such that $N(s_j) \cap Y_> \neq \emptyset$. Let X_u be the set of all internal non–support vertices s_j such that $N(s_j) \cap Y_> = \emptyset$. Then $X = X_s \cup X_u$ is the set of all internal non–support vertices of G.

Question 7.13 *Let* $G \cong C(p_1, p_2, \ldots, p_n)$, $n \in \mathbb{N}$, *with* $X = X_u \cup X_s \neq \emptyset$, *and let* $\ell_{\max} = \min(\Delta, k+1)$. *Is it true that*

(a) $\gamma_{\ell,k}(G) = \sum_{j=1;\ p_j \neq 0}^{n} \gamma_{\ell,k}(K_{1,p_j}) + \gamma_{\ell,k}(\langle X \rangle)$ *for any* $\ell \in \{1, 2, \ldots, \ell_{\max} - 1\}$ *and*

(b) $\gamma_{\ell_{\max},k}(G) = \sum_{j=1;\ p_j \neq 0}^{n} \gamma_{\ell_{\max},k}(K_{1,p_j}) + \gamma_{\ell_{\max},k}(\langle X_u \rangle)$

for any $k \in \mathbb{N}$? ■

Another class of trees considered in §6.6.3, is the class of spiders. Although the infinite order domination numbers were determined for this class, as stated in Proposition 6.31 and Corollary 6.13, only upper bounds for the smart finite order domination numbers were obtained. These bounds depend on the composition of the spider and are stated in Propositions 6.28–6.30. All of these bounds are expected to be sharp, providing the following open problem.

Question 7.14 *Let $S_{m\times n}$, $m \geq 2$, $n \geq 3$, be a spider and $k \in \mathbb{N}$, $\ell \in \{1, 2, \ldots, \ell_{\max} - 1\}$, with $\ell_{\max} = \min(\Delta, k + 1)$. Is it true that*

(a) $\gamma_{\ell_{\max},k}(S_{m\times n}) = \ell_{\max} + \Delta\gamma_{\ell_{\max},k}(P_{n-2})$ *if* $\gamma_{\ell_{\max},k}(P_{n-2}) < \gamma_{\ell_{\max},k}(P_{n-1})$*?*

(b) $\gamma_{\ell,k}(S_{m\times n}) = \Delta + \Delta\gamma_{\ell,k}(P_{n-2})$ *if* $\gamma_{\ell,k}(P_{n-2}) < \gamma_{\ell,k}(P_{n-1})$*?*

(c) $\gamma_{\ell,k}(S_{m\times n}) = 1 + \Delta\gamma_{\ell,k}(P_{n-2})$ *if* $\gamma_{\ell,k}(P_{n-2}) = \gamma_{\ell,k}(P_{n-1})$*?* ■

In Chapter 6, the special class of graphs known as hexagonal graphs was explored in §6.7, and the infinite domination numbers determined by Burger *et al.* [3]. The resemblance of higher order domination to a game of strategy was noted, and although the finite order parameter values are yet to be determined, properties of these parameters may have great significance regarding such games.

Question 7.15 *For the hexagonal graph $\mathcal{H}_{p,q}$, is it possible to determine the parameter values $\gamma_{\ell,k}(\mathcal{H}_{p,q})$ and $\gamma^*_{\ell,k}(\mathcal{H}_{p,q})$ for any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, k + 1\}$?* ■

Although some of the higher order domination numbers were explored for some special graph classes in Chapter 6, many other classes remain to be considered. Except for the open problems already encountered during the investigation of the special graph classes mentioned in this section, the following question attempts to suggest which other classes should be considered in the immediate future.

Question 7.16 *Is it possible to obtain both the higher order domination numbers for the following special graph classes?*

(a) The cartesian products $K_p \times P_q$, $K_p \times C_q$ and $C_p \times P_q$, for any $p, q \in \mathbb{N}$.

(b) The class of circulants. ■

As stated in §4.5, the NP–completeness of the decision problem $(\ell_{\max}, k)$–SMART DOMINATING FUNCTION, where no restriction is placed on the number of guards stationed at a vertex, was determined by Henning [18]. A decision problem (ℓ, k)–FOOLPROOF DOMINATING FUNCTION, concerning foolproof finite order domination, was also defined in §4.5 and was shown to belong to the class NP. The NP–completeness of this problem, however, has not yet been determinded.

Question 7.17 *For $\ell, k \in \mathbb{N}$, is it possible to prove that (ℓ, k)–FOOLPROOF DOMINATING FUNCTION $\in$ NP–complete?* ■

7.3 Suggested Generalised Protection Scenarios

There exist many different possible generalisations that extend the current framework for higher order domination provided by Definitions 4.1, 4.2, 5.1 and 5.2. In an attempt to arrive at a framework that caters for a more realistic protection setting, possible generalisations of the above mentioned definitions are suggested in this section, as required in thesis Objective II, stated in §1.3. The goal is to establish a cascade of conditions, incorporating more realistic characteristics of the higher order domination problem into the definition with each added condition, thereby hopefully ultimately allowing for the efficient modelling of real–world problems. Note that no formal definitions are provided, but only an informal discussion, to serve as suggestions for future generalisations. Also, the condition of smart and foolproof domination is disregarded in this discussion, since it has already been incorporated into the current definitions.

For the sake of this discussion, let the protection of a problem vertex by means of a guard move, be called an *attack round*. The protection of a graph against a sequence of attack rounds, be it of finite or infinite length, is considered in this section. A definition of some form of higher order domination may comprise a number of conditions concerning the graph being protected, the attack round characteristics, and the allowable guard movements. Four seemingly natural conditions are listed, as well as their complements. A generalised definition may contain each condition, or its complement.

Condition 1: The initial guard deployment, as well as the deployment after the protection of an attack round, is required to be a safe guard function (dominating set). This condition is included in the current definitions of Chapters 4 and 5. The complement to this condition is the scenario where there is no requirement that a redeployment be a safe guard function.

Condition 2: Multiple attacks may occur simultaneously during an attack round. This condition generalises the current notion of protection against only a single attack during an attack round, which may be viewed as the complement condition. Generalising a definition to cater for simultaneous attacks may have significant implications, as this allows for a much more realistic setting (as is, for example, the case in a typical board game of strategy).

Condition 3: Each attack round is required to be protected within a prespecified number of $m \geq 2$, say, moves. The complement of this condition, i.e. the protection against each attack round within one move, is currently catered for in the definitions of this thesis. Allowing for this condition may bring about significant benefits in terms of total number of guards required for the protection of the graph.

Condition 4: Guards may move simultaneously to protect against an attack round. Although this condition is not catered for in the current definitions of this thesis, it was introduced by Goddard *et al.* [12]. The complement of this condition, where only one guard may move to protect against an attack, is contained in the definitions of Chapters 4 and 5.

Note that it is meaningless to allow both Condition 2 and the complement of Condition 4 to hold simultaneously, since more than one guard will be required to protect the graph against more than one simultaneous attack. Also, if Condition 3 holds in conjunction with Condition 4, a *move* may be considered to count as one for all guards simultaneously.

If Condition x is abbreviated by $\mathbf{C}x$ and its complement by $\mathbf{C}'x$, then it is noted that the current definitions of this thesis may be abbreviated informally by $\{\mathbf{C1, C'2, C'3, C'4}\}$. Figure 7.3 illustrates the various generalisations by way of a rooted tree structure. Each decending path spanning the four levels, denotes a different definition. Only the half of the possible generalisations with **C1** included are shown, since the other half is similar in structure. Each of these generalisations incorporates different combinations of generalising conditions. The current definitions of this thesis, as mentioned above, is the most simplistic of these, while a definition corresponding to $\{\mathbf{C'1, C2, C3, C4}\}$ would be expected to be the most complex and realistic.

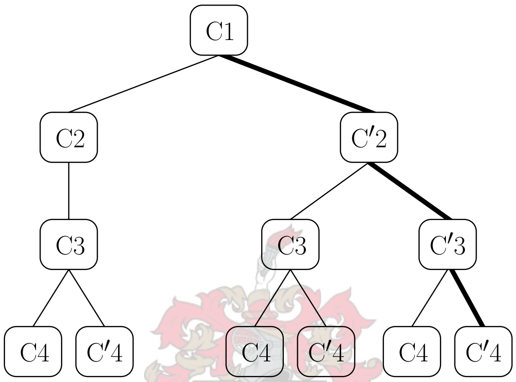


Figure 7.1: *Various suggested generalisations (necessarily including the condition* **C1***) on the current definition of higher order domination, represented by way of a rooted tree. The combination denoting the current definition, is indicated by the dark path.*

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In addition to Conditions 1–4 listed above, various other requirements may be introduced, depending on the practical problem being modelled. It may be required that the graph under consideration be a directed graph, thereby restricting the manner in which guards may move while protecting the graph. If the network structure has pathways that are navigated with various degrees of intensity, it may be significant to assign weights to the edges of the graph modelling this network. By introducing some movement value, the edges that guards may traverse while protecting a vertex, may be varied. Similarly, the vertices of a graph may also be weighted, so as to model the situation where some locations are more vulnerable to attack, or where some locations are more difficult to defend than others. Finally, it may be reasonable to assume, for a realistic scenario, that the guard defending a vertex will be weaker after its defense and hence it may not be available immediately to protect another vertex afterwards. Some sort of delay time parameter might be introduced to allow for such a situation, where a guard will have to wait a certain number of attack rounds before being able to defend another attacked vertex. These are just some of the generalisations that may be considered when attempting to create a framework wherein realistic defense–location problems are modelled and solved.

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Chapter A

Appendices

The purpose of this chapter is to provide the reader with additional information, as well as some useful results required for efficient proof structures in the main body of the thesis. The various sections in these appendices are ordered according to the order of reference in the main body of this document.

A.1 Additional Practical Motivation

Until the mid–19$^{\text{th}}$ century, Britain possessed a sufficiently strong naval force to deploy its so–called *battle fleets*, consisting of approximately twenty ships, over all important regions [25]. By the end of the century their strategy had to be revised, due to weakening British power, modernisation to steam propulsion, and the increasing strength of the German navy. Even though modernisation allowed fewer ships to constitute a battle fleet, Britain had only four battle fleets available by 1900. Similar to the situation of the Roman empire, two battle fleets were required to occupy a region before one could move to another region. The regions of interest in 1900, shown in Figure A.1, may be modelled by the graph shown in Figure A.2(a). The strategy decided upon by Lord John Fisher was to place three battle fleets in Britain and the remaining one in the Mediterranean. Only three of the six regions, namely Britain, the Mediterranean and the West Indies, were secured through this strategy, as shown in Figure A.2(a). Various different deployments of 4 battle fleets (13 different deployments, in fact, as shown in [25]), which is the minimum number needed to secure the area, are able to secure the entire graph shown in Figure A.2. An example would be to place two battle fleets in Britain and two in South Asia, as shown in Figure A.2(b). It has to be assumed that Fisher had sufficient reason to justify his choice of deployment, even though various different optimal deployment strategies existed.

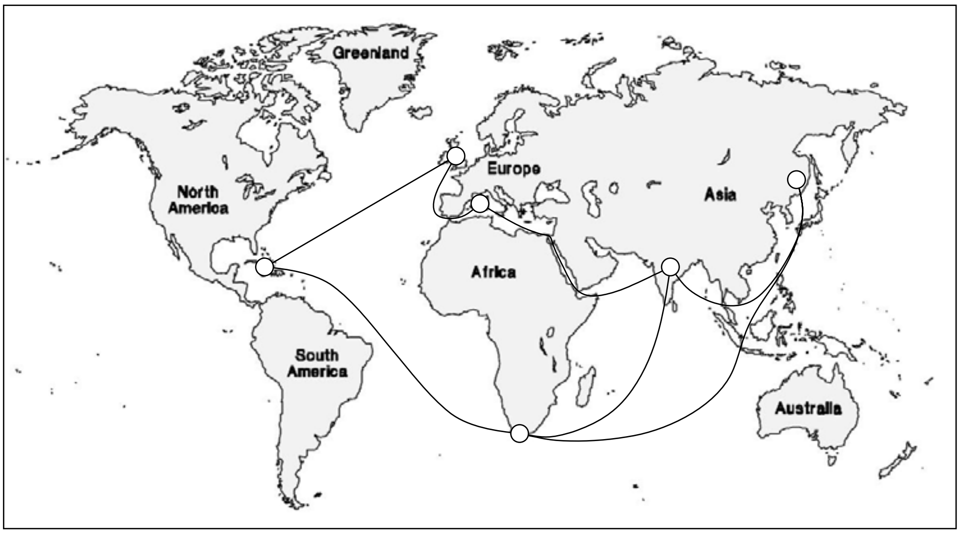

Figure A.1: *The regions of interest and their accessibility to the British navy around 1900. The regions are Britain, the Mediterranean, the Far East, South Asia, the Cape of Good Hope and the West Indies. (Original world map obtained from [21].)*

A.2 Properties of the Floor and Ceiling Operations

Proposition A.1

$$\lfloor a \rfloor - \lceil a \rceil = \begin{cases} 0 & \text{if } a \in \mathbb{Z} \\ -1 & \text{otherwise.} \end{cases}$$

Proof: If $a \in \mathbb{Z}$, the result follows trivially by definition of the floor and ceiling functions. Let $a \in \mathbb{R} \backslash \mathbb{Z}$ and $a = b + \varepsilon$, $0 < \varepsilon < 1$ and $b \in \mathbb{Z}$. Then $\lfloor a \rfloor - \lceil a \rceil = b - (b+1) = -1$. ■

The following three results may be useful in clarifying certain aspects of the proof of Theorem 6.1 and are mainly used in sections A.3 and A.4.

Proposition A.2 *For any $a \in \mathbb{Z}$ and $b \in \mathbb{R}$, $\lceil a + b \rceil = a + \lceil b \rceil$.*

Proof: Since the result is trivially true if $b \in \mathbb{Z}$, let $b = c + \varepsilon$, with $c \in \mathbb{Z}$ and $0 < \varepsilon < 1$. Then

$$\begin{aligned} \lceil a + b \rceil &= \lceil a + c + \varepsilon \rceil \\ &= a + c + 1 \\ &= a + \lceil c + 1 \rceil \\ &= a + \lceil c + \varepsilon \rceil \\ &= a + \lceil b \rceil, \end{aligned}$$

which is the desired result. ■

(a) *A graph modelling the area of Figure A.1, with the deployment strategy followed by Lord Fisher indicated by dark vertices.*

(b) *An example of a minmimum weight Roman dominating function for the graph.*

Figure A.2: *(a) A graph of nodes and inter–connecting lines used to model the regions of interest to the British navy around 1900. The various regions are: $v_1 \equiv$ Britain, $v_2 \equiv$ Mediterranean, $v_3 \equiv$ Far East, $v_4 \equiv$ South Asia, $v_5 \equiv$ Cape of Good Hope, $v_6 \equiv$ West Indies. The strategy followed by Lord Fisher is indicated as dark vertices, while the vertex with more than one guard stationed at it is indicated by the appropriate numerical value. (b) An example of a minimum weight Roman dominating function of the graph, with occupied vertices indicated accordingly.*

Proposition A.3 *For any $a \in \mathbb{R}\backslash\mathbb{Z}$, $\lceil -a \rceil = -\lceil a \rceil + 1$.*

Proof: Let $a = b + \varepsilon$, with $b \in \mathbb{Z}$ and $0 < \varepsilon < 1$. Then

$$\begin{aligned} -\lceil -a \rceil &= -\lceil -b - \varepsilon \rceil \\ &= -(-b) \\ &= b + 1 - 1 \\ &= \lceil b + 1 \rceil - 1 \\ &= \lceil b + \varepsilon \rceil - 1 \\ &= \lceil a \rceil - 1, \end{aligned}$$

and the result follows. ■

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Corollary A.1 *For any $a \in \mathbb{Z}$ and $b \in \mathbb{R}\backslash\mathbb{Z}$, $a - \lceil b \rceil = \lceil a - b \rceil - 1$.*

Proof: Using Proposition A.2 and A.3, it follows that

$$\begin{aligned} a - \lceil b \rceil &= -(-a + \lceil b \rceil) \\ &= -(\lceil -a + b \rceil) \\ &= \lceil -(-a + b) \rceil - 1 \\ &= \lceil a - b \rceil - 1, \end{aligned}$$

which is the desired result. ■

Proposition A.4 *For any $a, \varepsilon \in \mathbb{R}$, with $0 \le \varepsilon \le 1$, $\lceil a \rceil - 1 \le \lceil a - \varepsilon \rceil$.*

Proof: Let $a = c + \delta$, with $c \in \mathbb{Z}$ and $0 \le \delta < 1$. Then $\lceil a \rceil - 1 = \lceil c + \delta \rceil - 1 = c$. Since $\lceil a - \varepsilon \rceil = \lceil c + \delta - \varepsilon \rceil = c + 1$ or c, the inequality follows. ■

The following result is useful for the proof of Proposition 6.9.

Proposition A.5 *For any $p, q \in \mathbb{N}$, $\lceil \frac{p}{2} \rceil \lceil \frac{q}{2} \rceil + \lfloor \frac{p}{2} \rfloor \lfloor \frac{q}{2} \rfloor = \lceil \frac{pq}{2} \rceil$.*

Proof: If p or q, say p, is even, then

$$\left\lceil \frac{p}{2} \right\rceil \left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor = \frac{p}{2} \left(\left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{q}{2} \right\rfloor \right) = \frac{pq}{2}.$$

If both p and q are odd, then

$$\left\lceil \frac{p}{2} \right\rceil \left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor = \frac{p+1}{2} \cdot \frac{q+1}{2} + \frac{p-1}{2} \cdot \frac{q-1}{2} = \frac{pq+1}{2}.$$

The result follows from this. ■

The following result is useful for the proof of Theorem 6.3.

Proposition A.6 *If $x \le b\lceil \frac{a}{b} \rceil - a$, then $x\lfloor \frac{a}{b} \rfloor + (b - x)\lceil \frac{a}{b} \rceil \ge a$ for any $a, b \in \mathbb{Z}$, with equality if $x = b\lceil \frac{a}{b} \rceil - a$.*

Proof: If $\frac{a}{b} \in \mathbb{Z}$, then

$$x \left\lfloor \frac{a}{b} \right\rfloor + (b - x) \left\lceil \frac{a}{b} \right\rceil = b \left\lceil \frac{a}{b} \right\rceil = a.$$

If $\frac{a}{b} \notin \mathbb{Z}$, then by Proposition A.1,

$$x \left\lfloor \frac{a}{b} \right\rfloor + (b - x) \left\lceil \frac{a}{b} \right\rceil = b \left\lceil \frac{a}{b} \right\rceil - x \ge a,$$

with equality if $x = b\lceil \frac{a}{b} \rceil - a$. ■

A.3 Derivation of Equation (6.1) in Theorem 6.1

Suppose the path $P_n : w_1, \ldots, w_n$ is partitioned into $\lfloor \frac{n}{4k+3} \rfloor$ subpaths $P_{4k+3}^{(t)}$ of order $4k + 3$ ($t = 1, \ldots, \lfloor \frac{n}{4k+3} \rfloor$) and one (possibly empty) subpath $P_c : y_1, \ldots, y_c$ of order $c \equiv n \pmod{4k + 3}$. It will be shown that

$$\left\lfloor \frac{n}{4k+3} \right\rfloor (2k+1) + \left\lceil \frac{c}{2} \right\rceil = \left\lceil \frac{2k+1}{4k+3} n \right\rceil. \tag{A.1}$$

From the above partition of P_n, it follows that

$$n = \left\lfloor \frac{n}{4k+3} \right\rfloor (4k+3) + c.$$

Therefore

$$\frac{2k+1}{4k+3} n = \left\lfloor \frac{n}{4k+3} \right\rfloor (2k+1) + \frac{2k+1}{4k+3} c,$$

and hence

$$\begin{aligned} \left\lceil \frac{2k+1}{4k+3} n \right\rceil &= \left\lceil \left\lfloor \frac{n}{4k+3} \right\rfloor (2k+1) + \frac{2k+1}{4k+3} c \right\rceil \\ &= \left\lfloor \frac{n}{4k+3} \right\rfloor (2k+1) + \left\lceil \frac{2k+1}{4k+3} c \right\rceil, \end{aligned} \tag{A.2}$$

by Proposition A.2.

Furthermore,

$$\begin{aligned} \left\lceil \frac{2k+1}{4k+3} c \right\rceil &= \left\lceil \frac{2k + \frac{3}{2} - \frac{1}{2}}{4k+3} c \right\rceil \\ &= \left\lceil \frac{c}{2} - \frac{1}{2} \frac{c}{4k+3} \right\rceil \\ &= \left\lceil \frac{c}{2} \right\rceil, \end{aligned}$$

since $c < 4k+3$ implies that $\frac{1}{2}\frac{c}{4k+3} < \frac{1}{2}$. Therefore (A.2) is precisely equality (6.1) in Theorem 6.1.

A.4 Derivation of Inequality (6.3) in Theorem 6.1

Let $k \leq n-2$ and partition the path $P_n : w_1, \ldots, w_n$ into $\lfloor \frac{n}{k+3} \rfloor$ subpaths $P_{k+3}^{(j)}$: $w_{j(k+3)+1}, w_{j(k+3)+2}, \ldots, w_{j(k+3)+k+3}$, $j = 0, \ldots, \lfloor \frac{n}{k+3} \rfloor - 1$ and one (possibly empty) subpath $P_c : w_{\lfloor n/(k+3) \rfloor (k+3)+1}, \ldots, w_n$ of order $c \equiv n \pmod{k+3}$. Consider the safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$, where

$$V(P_{k+3}^{(j)}) \cap V_1^{(0)} = \{w_i : i \equiv 2, \ldots, k+2 \pmod{k+3}\}, \quad j = 0, \ldots, \left\lfloor \frac{n}{k+3} \right\rfloor - 1,$$

where

$$V(P_c) \cap V_1^{(0)} = \begin{cases} \{w_i : \ i \equiv 1, 2, \ldots, c \pmod{k+3}\} & \text{if } 1 \leq c \leq \lfloor (k+2)/2 \rfloor \\ \{w_i : \ i \equiv 2, 3, \ldots, c \pmod{k+3}\} & \text{if } \lfloor (k+2)/2 \rfloor < c < k+3 \end{cases}$$

and where $V_0^{(0)} = V(P_n) \backslash V_1^{(0)}$. Clearly $f^{(0)}$ is a $(1,k)$–FDF for P_n. Also note that

$$\left\lceil \frac{2c+1}{k+3} \right\rceil = \begin{cases} 1 \text{ if } 1 \leq c \leq \left\lfloor \frac{k+2}{2} \right\rfloor \\ 2 \text{ if } \left\lfloor \frac{k+2}{2} \right\rfloor < c < k+3. \end{cases}$$

If $\frac{2c+1}{k+3} \in \mathbb{Z}$, then

$$\begin{aligned}
\gamma^*_{1,k}(P_n) &\leq w(f^{(0)}) \\
&= \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + (c+1) - \left\lceil \frac{2c+1}{k+3} \right\rceil \\
&= \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil c+1 - \frac{2c+1}{k+3} \right\rceil \\
&= \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil c+1 - \frac{2c+1}{k+3} - \frac{k+2}{k+3} \right\rceil \\
&= \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil \frac{k+1}{k+3} c \right\rceil,
\end{aligned}$$

by utilisation of Proposition A.2. If $\frac{2c+1}{k+3} \in \mathbb{R} \backslash \mathbb{Z}$, then

$$\begin{aligned}
\gamma^*_{1,k}(P_n) &\leq w(f^{(0)}) \\
&= \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + (c+1) - \left\lceil \frac{2c+1}{k+3} \right\rceil \\
&= \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil c+1 - \frac{2c+1}{k+3} \right\rceil - 1 \\
&\leq \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil c+1 - \frac{2c+1}{k+3} - \frac{k+2}{k+3} \right\rceil \\
&= \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil \frac{k+1}{k+3} c \right\rceil,
\end{aligned}$$

by utilising firstly Proposition A.3 and then Proposition A.4.

From the above two cases, it follows that

$$\gamma^*_{1,k}(P_n) \leq \left\lfloor \frac{n}{k+3} \right\rfloor (k+1) + \left\lceil \frac{k+1}{k+3} c \right\rceil,$$

which is precisely inequality (6.3) in Theorem 6.1.

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